Stationary Heston model: Calibration and Pricing of exotics using Optimal Quantization

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Stationary Heston model

The model

Dynamic of the asset price process $(S_t^{(\nu)})_{t\geqslant 0}$ and its volatility $(v_t^{\nu})_{t\geqslant 0}$ is given by

$$\begin{cases} \dfrac{dS_t^{(\nu)}}{S_t^{(\nu)}} = (r-q)dt + \sqrt{v_t^{\nu}}\left(\rho d\widetilde{W}_t + \sqrt{1-\rho^2}dW_t\right) \\ dv_t^{\nu} = \kappa(\theta - v_t^{\nu})dt + \xi\sqrt{v_t^{\nu}}d\widetilde{W}_t \end{cases}$$

- $S_0^{(\nu)} = s_0$ is the initial value of the process, r the spot rate, q the dividend rate,
- κ the mean reverting term,
- θ the long run average price variance,
- ξ is the volatility of the volatility,
- $(W, \widetilde{W})$ is a standard correlated 2d Brownian motion with correlation ρ,

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- ξ is the volatility of the volatility,
- $(W, \widetilde{W})$ is a standard correlated 2d Brownian motion with correlation ρ,
- **$v_0^{\nu} \sim \mathcal{L}(\nu) \sim \Gamma(\alpha, \beta)$ with $\beta = (2\kappa)/\xi^2$ and $\alpha = \theta\beta$.**

Remark: 4 parameters $\Longrightarrow$ 1 less than the Standard Heston Model.

History

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Generic expression

The price of the European option on the asset $S_T^{(\nu)}$ is given by

$$I_0 = \mathbb{E}\left[\, \mathrm{e}^{-rT}\, \varphi(S_T^{(\nu)})\right].$$

After preconditioning by v_0^{ν}, we have

$$I_0 = \mathbb{E}\left[\, \mathbb{E}\left[\, \mathrm{e}^{-rT}\, \varphi(S_T^{(\nu)}) \mid \sigma(v_0^{\nu})\right]\right] = \mathbb{E}\left[f(v_0^{\nu})\right]$$

where $f(v)$ is the price of the European option in the Standard Heston model with deterministic initial conditions for the set of parameters $\lambda(v) = (s_0, r, q, \theta, \kappa, \xi, \rho, v)$.

Example - Call option

If φ is the payoff of a Call option then f is simply the price given by Fourier transform in the standard Heston model of the European Call Option. Then

$$I_0 = \mathbb{E}\left[\mathrm{e}^{-rT}(S_T^{(\nu)} - K)_+\right] = \mathbb{E}\Big[C\big(\lambda(v_0^\nu), K, T\big)\Big]$$

with

$$C(\lambda(v), K, T) = S_0^{(v)}\,\mathrm{e}^{-qT}\,P_1\big(\lambda(v), K, T\big) - K\,\mathrm{e}^{-rT}\,P_2\big(\lambda(v), K, T\big)$$

and

$$P_1\big(\lambda(v), K, T\big) = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \mathrm{Re}\left(\frac{\mathrm{e}^{-\mathbf{i}u\log(K)}}{iu}\,\frac{\psi\big(\lambda(v), u-\mathbf{i}, T\big)}{s_0\,\mathrm{e}^{(r-q)T}}\right)du$$

$$P_2\big(\lambda(v), K, T\big) = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \mathrm{Re}\left(\frac{\mathrm{e}^{-\mathbf{i}u\log(K)}}{\mathbf{i}u}\psi\big(\lambda(v), u, T\big)\right)du$$

where $\lambda(v) = \big(s_0, r, q, \theta, \kappa, \xi, \rho, v\big)$.

Practical aspects

Fixed-point quadratures

- I_0 can be written as an integral against the Laguerre weighting function

$$I_0 = \int_0^{+\infty} f(v) \frac{\beta^\alpha}{\Gamma(\alpha)} v^{\alpha-1} \mathrm{e}^{-\beta v}\, dv = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} f(v)\omega(v) dv$$

where $\omega(v) = v^{\alpha-1}\mathrm{e}^{-\beta v}$ is the Laguerre weighting function.

- Then, for a fixed $n > 0$ with ω_i's and v_i's the associated Laguerre weights and nodes, I_0 would be approximated by

$$\widetilde{I}_0^n = \frac{\beta^\alpha}{\Gamma(\alpha)} \sum_{i=1}^{n} \omega_i f(v_i).$$

Quantization-based cubature method

Approximate I_0 using the following quantization-based cubature formula

$$\widehat{I}_0^N = \mathbb{E}\left[f(\widehat{v}_0^N)\right] = \sum_{i=1}^{N} f(v_{0,i}^N)\, \mathbb{P}(\widehat{v}_0^N = v_{0,i}^N).$$

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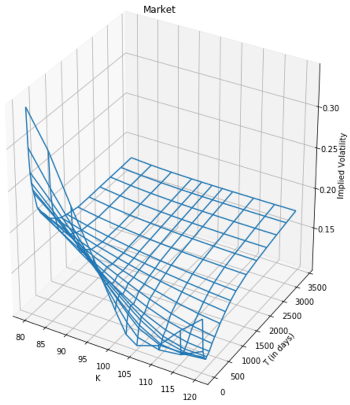
The implied volatility surface from the market

Euro Stoxx 50 - 26th September 2019: $S_0 = 3541$, $r = -0.32\%$, $q = 0.225\%$



Let

$$\mathcal{P}_{SH} = \left\{(\theta, \kappa, \xi, \rho) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times [-1, 1]\right\}$$

the calibrable Stationary Heston parameters and

$$\mathcal{P}_{H} = \left\{(x, \theta, \kappa, \xi, \rho) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times [-1, 1]\right\}$$

the calibrable Standard Heston parameters.

The problem (without penalization)

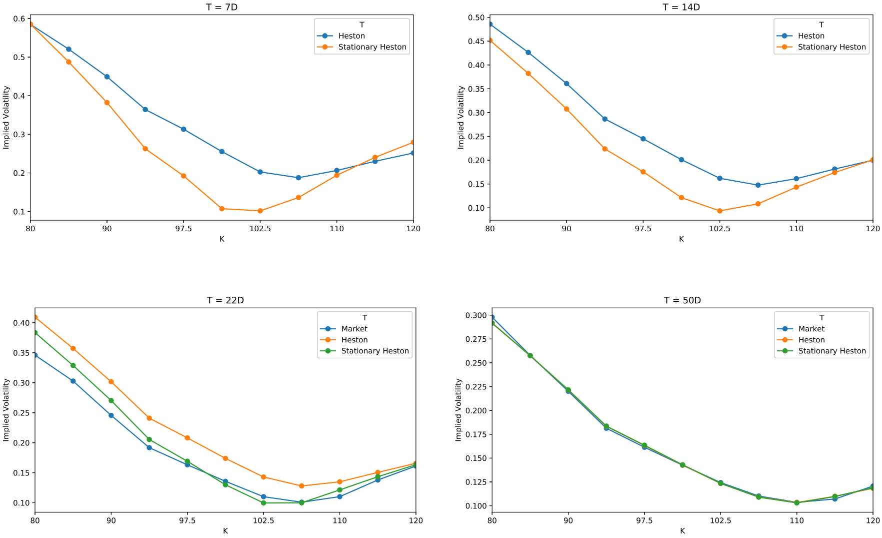
Find $\phi^\star$ that minimizes

$$\min_{\phi \in \mathcal{P}} \sum_K \left(\frac{\sigma_{\text{iv}}^{Market}(K, T) - \sigma_{\text{iv}}^{Model}(\phi, K, T)}{\sigma_{\text{iv}}^{Market}(K, T)} \right)^2$$

where

- $\sigma_{\text{iv}}^{Market}(K, T)$ is the implied volatility deduced from the market,
- $\sigma_{\text{iv}}^{Model}(\phi, K, T)$ is the implied volatility of the EU Call/Put price computed with a chosen model (Stationary Heston or Standard Heston).

Calibration to expiry 50 days ($T = 50/365$). Relative calibration errors: $< 3\%$ for each implied volatility.



The standard Heston model fails to produce the desired smile for very small maturities while the Stationary model has no problem to generate it with 1 parameter less.

Term-structure comparison

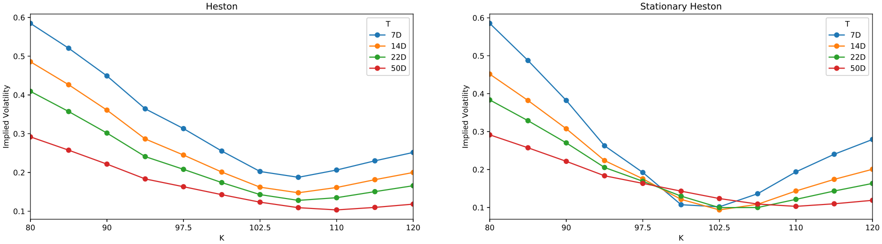
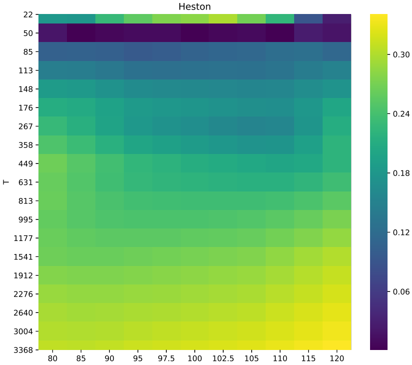


Figure: Term-structure of the volatility in function of T and K of both models (left: Standard Heston and right: Stationary Heston) after calibration at 50 days.

Relative Errors comparison



The obtained parameters

$\phi^\star$	ρ	v_0	θ	κ	ξ
Standard Heston	-0.74	0.152584	0.01487	80.05	5.22
Stationary Heston	-0.75		0.02744	593.46	36.80

Figure: *Parameters obtained for both models after calibration without penalization for options with maturity* 50 *days.*

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Figure: *Parameters obtained for both models after calibration without penalization for options with maturity* 50 *days.*

- They do not fulfil the Feller condition

$$\xi^2 \leqslant 2\kappa\theta.$$

Hence, we cannot simulate the model without reaching 0.

New calibration problem (with penalization)

Find $\phi^\star$ that minimizes

$$\min_{\phi\in\mathcal{P}} \sum_K \left(\frac{\sigma_{\text{iv}}^{Market}(K,T) - \sigma_{\text{iv}}^{Model}(\phi,K,T)}{\sigma_{\text{iv}}^{Market}(K,T)} \right)^2 + \lambda \max(\xi^2 - 2\kappa\theta, 0)$$

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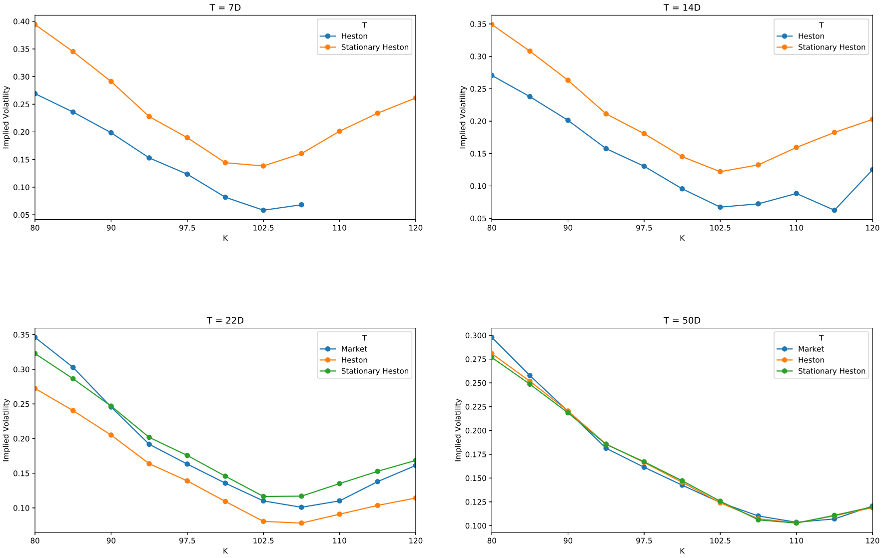
- Obtained parameters:

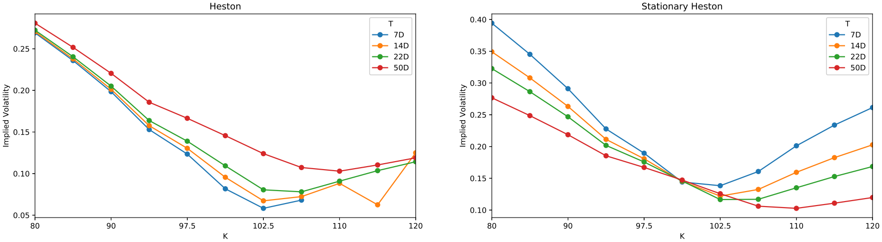
$\phi^\star$	ρ	v_0	θ	κ	ξ
Standard Heston	-0.83	0.0045	0.17023	2.19	1.04
Stationary Heston	-0.99		0.02691	19.28	1.15

Figure: *Parameters obtained for both models after calibration with penalization ($\lambda = 0.01$) for options with maturity 50 days.*

Calibration to expiry 50 days ($T = 50/365$). Relative calibration errors: $< 3\%$ for each implied volatility.



Term-structure comparison



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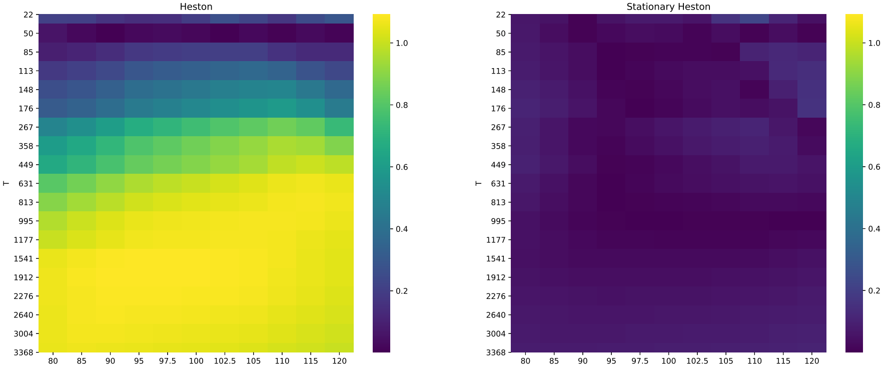

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Definitions

Let $\Gamma_N = \{x_1^N, \ldots, x_N^N\} \subset \mathbb{R}$, a subset of size N, called N-quantizer, we define

- The *Voronoï partition* of $\mathbb{R}$ induced by the N-quantizer

$$C_i(\Gamma_N) = \big(x_{i-1/2}^N, x_{i+1/2}^N\big], \quad i \in [\![1, N-1]\!], \quad C_N(\Gamma_N) = \big(x_{N-1/2}^N, x_{N+1/2}^N\big).$$

  Easily defined in dimension one.

- The *Voronoï Quantization* of the random variable X

$$\widehat{X}^{\Gamma_N} = \mathrm{Proj}_{\Gamma_N}(X) = \sum_{i=1}^{N} x_i^N \, \mathbb{1}_{X \in C_i(\Gamma_N)}$$

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- It is convenient to define the quadratic distortion function at level N

$$\mathcal{Q}_{2,\mathrm{N}} : x = (x_1^N, \dots, x_N^N) \longmapsto \mathbb{E}\Big[\min_{i \in [\![1,N]\!]} |X - x_i^N|^2 \Big] = \|X - \widehat{X}^N\|_2^2.$$

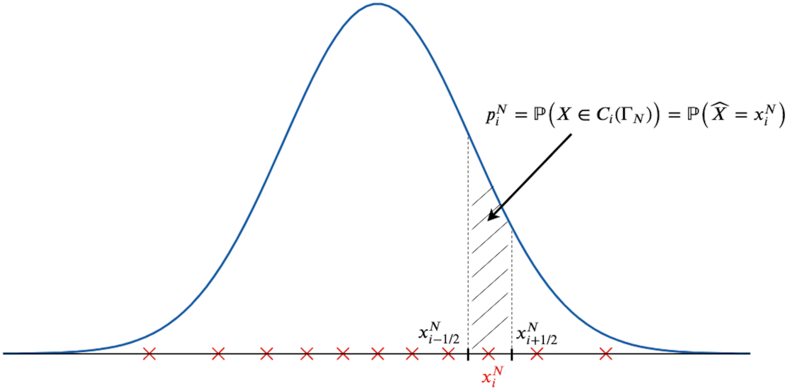

Figure: Gaussian Optimal Quantization

How to build an Optimal Quantizer?

1. Differentiate the $\mathcal{Q}_{2,\mathrm{N}}$

The gradient is given by

$$\nabla \mathcal{Q}_{2,\mathrm{N}}(x_{\mathbf{1}:N}) = \left(\mathbb{E}\left[(x_i^N - X)\, \mathbb{1}_{X \in \left(x^N_{i-\mathbf{1/2}}, x^N_{i+\mathbf{1/2}}\right]} \right] \right)_{i=1,\dots,N}$$

2. Solve the fixed point problem

Find $x_{1:N}$ that cancel the gradient

$$\nabla \mathcal{Q}_{2,\mathrm{N}}\left(x_{\mathbf{1}:N}\right) = 0 \quad \Longleftrightarrow \quad x_i^N = \frac{\mathbb{E}\left[X\, \mathbb{1}_{X \in \left(x^N_{i-\mathbf{1/2}}, x^N_{i+\mathbf{1/2}}\right]} \right]}{\mathbb{P}\left(X \in \left(x^N_{i-1/2}, x^N_{i+1/2}\right] \right)}, \qquad i = 1, \dots, N$$

$$\Longleftrightarrow \quad x_i^N = \frac{K_x\left(x^N_{i+1/2}\right) - K_x\left(x^N_{i-1/2}\right)}{F_x\left(x^N_{i+1/2}\right) - F_x\left(x^N_{i-1/2}\right)}, \qquad i = 1, \dots, N.$$

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Litterature

Recursive Quantization

- *Recursive marginal quantization of the Euler scheme of a diffusion process* by G. Pagès and A. Sagna. (2015)
- *Recursive Marginal Quantization of Higher-Order Schemes* by J. Kienitz, T. A. McWalter, E. Platen and R. Rudd. (2017)
- *Product Markovian quantization of an R^d-valued Euler scheme of a diffusion process with applications to finance* by L. Fiorin, G. Pagès and A. Sagna. (2018)

Previous work on Heston model using Quantization

- *Pricing via Quantization in Stochastic Volatility Models* by G. Callegaro, L. Fiorin and M. Grasselli. (2016)
- *Fast Quantization of Stochastic Volatility Models* by J. Kienitz, T. A. McWalter, E. Platen and R. Rudd. (2017)
- *American quantized calibration in stochastic volatility* by G. Callegaro, L. Fiorin and M. Grasselli. (2018)
- And more...

Recursive Quantization: the idea (1/3)

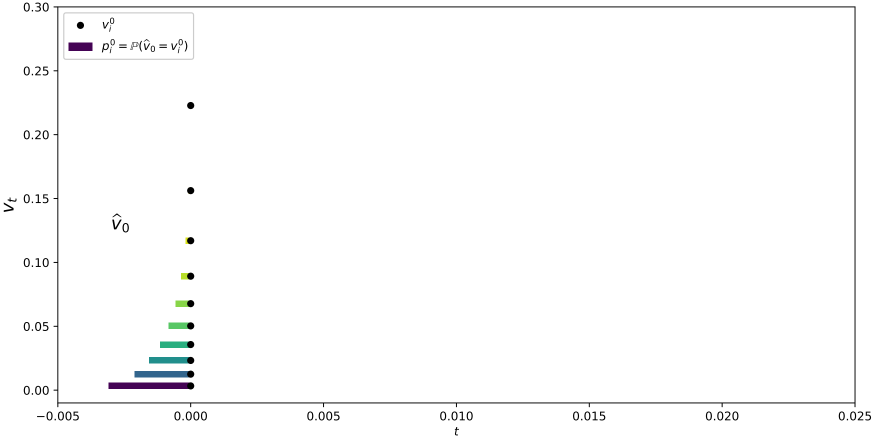

Figure: *Start from a grid at time t_0.*

Recursive Quantization: the idea (2/3)

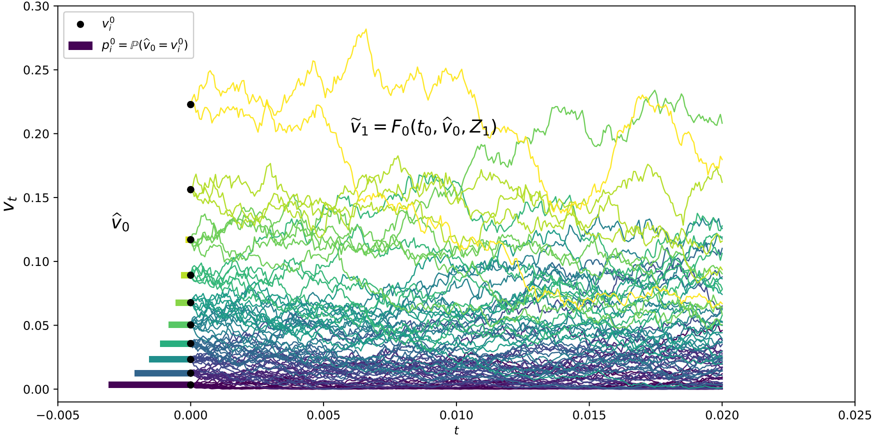

Figure: *Simulation from t_0 to t_1 with a given discretization scheme F_0.*

Recursive Quantization: the idea (3/3)

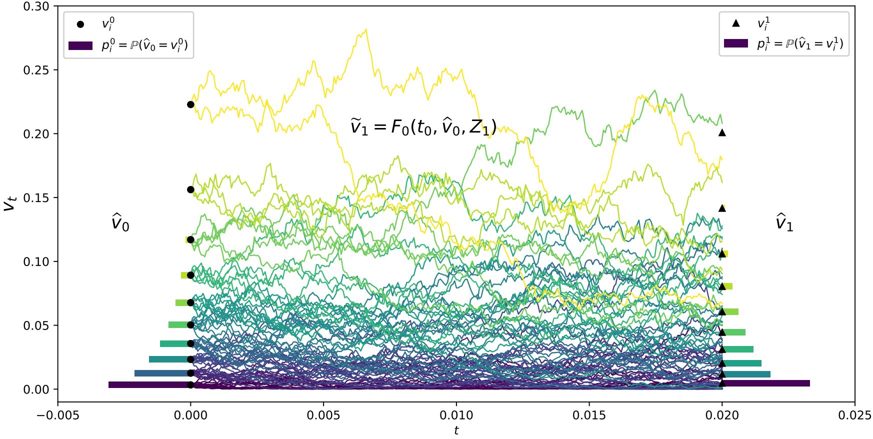

Figure: *Build the quantizer at time t_1.*

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Model transformation

$$\begin{cases} \dfrac{dS_t^{(\nu)}}{S_t^{(\nu)}} = (r-q)dt + \sqrt{v_t^{\nu}}\left(\rho d\widetilde{W}_t + \sqrt{1-\rho^2}dW_t\right) \\ dv_t^{\nu} = \kappa(\theta - v_t^{\nu})dt + \xi\sqrt{v_t^{\nu}}d\widetilde{W}_t \end{cases}$$

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We will be working with (X_t, Y_t) defined by

- For the volatility $\longrightarrow Y_t = e^{\kappa t}\, v_t^\nu$.
- For the asset $\longrightarrow X_t = \log(S_t^{(\nu)})$.

First, the volatility

Milstein Scheme (preserving the positivity)

We consider the following *boosted* volatility process: let $Y_t = e^{\kappa t} v_t^{\nu}, t \in [0, T]$.

$$dY_t = e^{\kappa t} \kappa\theta dt + \xi e^{\frac{\kappa t}{2}} \sqrt{Y_t} d\widetilde{W}_t.$$

Now, if we look at the Milstein discretization scheme of Y_t

$$\bar{Y}_{t_{k+1}} = \mathcal{M}_{\widetilde{b},\widetilde{\sigma}}\big(t_k, \bar{Y}_{t_k}, \widetilde{Z}_{k+1}\big)$$

where $\widetilde{Z}_{k+1} \sim \mathcal{N}(0,1)$ and

$$\mathcal{M}_{\widetilde{b},\widetilde{\sigma}}(t,x,z) = x - \frac{\widetilde{\sigma}(t,x)}{2\widetilde{\sigma}'_x(t,x)} + h\left(\widetilde{b}(t,x) - \frac{\widetilde{\sigma}\widetilde{\sigma}'_x(t,x)}{2}\right) + \frac{\widetilde{\sigma}\widetilde{\sigma}'_x(t,x)h}{2}\left(z + \frac{1}{\sqrt{h}\widetilde{\sigma}'_x(t,x)}\right)^2$$

with $h = T/n$, n the number of time-steps and

$$\widetilde{b}(t,x) = e^{\kappa t} \kappa\theta, \qquad \widetilde{\sigma}(t,x) = \xi\sqrt{x}\, e^{\frac{\kappa t}{2}} \qquad \text{and} \qquad \widetilde{\sigma}'_x(t,x) = \frac{\xi e^{\frac{\kappa t}{2}}}{2\sqrt{x}}.$$

Then, the log-asset

Euler-Maruyama scheme

We consider the logarithm of the asset $X_t = \log(S_t^{(\nu)})$, yielding

$$dX_t = \left(r - \frac{v_t}{2}\right) dt + \sqrt{v_t} dW_t.$$

Now, using an Euler-Maruyama scheme for the discretization of X_t, we have

$$\begin{cases} \bar{X}_{t_{k+1}} = \mathcal{E}_{b,\sigma}\left(t_k, \bar{X}_{t_k}, \bar{Y}_{t_k}, Z_{k+1}\right) \\ \bar{Y}_{t_{k+1}} = \mathcal{M}_{\widetilde{b},\widetilde{\sigma}}\left(t_k, \bar{Y}_{t_k}, \widetilde{Z}_{k+1}\right) \end{cases}$$

where $Z_{k+1} \sim \mathcal{N}(0,1)$ and $\mathbb{C}\mathrm{orr}(Z_{k+1}, \widetilde{Z}_{k+1}) = \rho$ and

$$\mathcal{E}_{b,\sigma}\left(t, x, y, z\right) = x + b(t,x,y)h + \sigma(t,x,y)\sqrt{h}\, z$$

with

$$b(t,x,y) = r - \frac{e^{-\kappa t} y}{2} \qquad \text{and} \qquad \sigma(t,x,y) = \sqrt{e^{-\kappa t} y}.$$

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First, the volatility

We build recursively the Markovian quantization tree $(\widehat{Y}_k)_{k \in [\![0,n]\!]}$ where $\widehat{Y}_{k+1}$ is the Voronoï quantization of $\widetilde{Y}_{k+1}$ defined by

$$\widetilde{Y}_{k+1} = \mathcal{M}_{\widetilde{b},\widetilde{\sigma}}\big(t_k, \widehat{Y}_k, \widetilde{Z}_{k+1}\big), \qquad \widehat{Y}_{k+1} = \mathrm{Proj}_{\Gamma^Y_{N_2}}\big(\widetilde{Y}_{k+1}\big)$$

with $\Gamma^Y_{N_2} = \{y_1^{k+1}, \dots, y_{N_2}^{k+1}\}$ the optimal N_2-quantizer of $\widetilde{Y}_{k+1}$ and $\widetilde{Z}_{k+1} \sim \mathcal{N}(0,1)$.

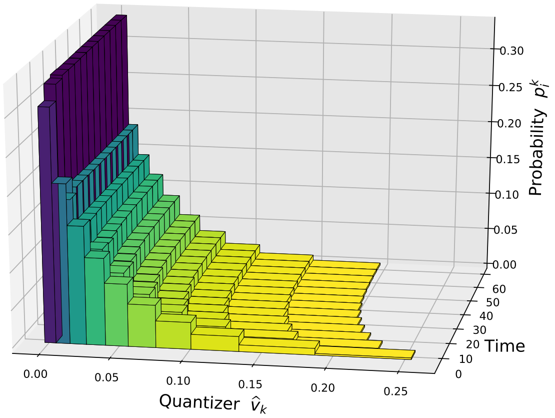

Figure: *Rescaled Recursive quantization of the boosted-volatility process with its associated weights from $t = 0$ to $t = 60$ days with a time step of 5 days with grids of size $N = 10$.*

Then, the log-asset

Now, using the fact that $(Y_t)_t$ has already been quantized and the Euler-Maruyama scheme of $(X_t)_t$, we define the Markov quantized scheme

$$\widetilde{X}_{k+1} = \mathcal{E}_{b,\sigma}\big(t_k, \widehat{X}_k, \widehat{Y}_k, Z_{k+1}\big), \qquad \widehat{X}_{k+1} = \mathrm{Proj}_{\Gamma^X_{N_1}}\big(\widetilde{X}_{k+1}\big)$$

with $\Gamma^X_{N_1} = \{x_1^k, \ldots, x_{N_1}^k\}$ the optimal N_1-quantizer of $\widetilde{X}_{k+1}$ and $Z_{k+1} \sim \mathcal{N}(0,1)$.

L^2-error

Notations

$\widehat{U}_k = (\widehat{X}_k, \widehat{Y}_k)$ is the product recursive quantization of $\bar{U}_k = (\bar{X}_k, \bar{Y}_k)$, the time-discretized processes defined by

$$\bar{U}_k = F_{k-1}(\bar{U}_{k-1}, Z_k), \quad \text{with} \quad F_k(u, Z) = \begin{pmatrix} \mathcal{E}_{b,\sigma}(t_k, x, y, Z^1_{k+1}) \\ \mathcal{M}_{\tilde{b},\tilde{\sigma}}(t_k, y, Z^2_{k+1}) \end{pmatrix}.$$

where $Z_k = (Z^1_k, Z^2_k)$ is a standardized correlated Gaussian vector.

What about the error induced by the recursive quantization?

L^2-error

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where $Z_k = (Z^1_k, Z^2_k)$ is a standardized correlated Gaussian vector.

What about the error induced by the recursive quantization?

Standard results of the type

$$\|\widehat{U}_k - \bar{U}_k\|_2 \leqslant \sum_{j=1}^{k} C_j \big(N_{1,j} \times N_{2,j}\big)^{-1/2}$$

when the schemes $F_k(u, z)$ are Lipschitz in u.

L^2-error

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when the schemes $F_k(u, z)$ are Lipschitz in u.
But this is not our case... (CIR model)

L^2-error

Proposition

For every $k = 0, \dots, n$

$$\|\widehat{U}_k - \bar{U}_k\|_2 \leqslant \sum_{j=0}^{k} \widetilde{A}_{j,k}\big(N_{1,j} \times N_{2,j}\big)^{-1/2} + B_k\sqrt{h}$$

where

$$\widetilde{A}_{j,k} = 2^{\frac{p-2}{2p}} C_p^2 A_{j,k}\left(2^{(\frac{p}{2}-1)j}\beta_p^j\|\widehat{U}_0\|_2^p + \alpha_p \frac{1-2^{(\frac{p}{2}-1)j}\beta_p^j}{1-2^{\frac{p}{2}-1}\beta_p}\right)^{1/p}$$

with

$$A_{j,k} = 2^{\frac{k-j}{2}}\,\mathrm{e}^{\frac{\sqrt{h}}{2}(k-j)} \quad \text{and} \quad B_k = C_T(h)\sum_{j=0}^{k-1} 2^{\frac{k-1-j}{2}}\,\mathrm{e}^{\frac{\sqrt{h}}{2}(k-1-j)}$$

where $\sum_{\varnothing} = 0$ *by convention and* $C_T(h) = O(1)$.

L^2-error

Proposition

For every $k = 0, \ldots, n$

$$\|\widehat{U}_k - \bar{U}_k\|_2 \leqslant \sum_{j=0}^{k} \widetilde{A}_{j,k}\left(N_{1,j} \times N_{2,j}\right)^{-1/2} + B_k\sqrt{h}$$

where

$$\widetilde{A}_{j,k} = 2^{\frac{p-2}{2p}} C_p^2 A_{j,k}\left(2^{(\frac{p}{2}-1)j}\beta_p^j\|\widehat{U}_0\|_2^p + \alpha_p \frac{1-2^{(\frac{p}{2}-1)j}\beta_p^j}{1-2^{\frac{p}{2}-1}\beta_p}\right)^{1/p}$$

with

$$A_{j,k} = 2^{\frac{k-j}{2}} e^{\frac{\sqrt{h}}{2}(k-j)} \quad \textit{and} \quad B_k = C_T(h)\sum_{j=0}^{k-1} 2^{\frac{k-1-j}{2}} e^{\frac{\sqrt{h}}{2}(k-1-j)}$$

where $\sum_{\varnothing} = 0$ *by convention and* $C_T(h) = O(1)$.

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Bermudan Options

Its price, at time t_0, is given by

$$V_0 = \sup_{\tau \in \{t_1, \cdots, t_n\}} \mathbb{E}\left[\, e^{-r\tau}\, \psi_\tau(X_\tau, Y_\tau) \mid \mathcal{F}_{t_0} \,\right].$$

Hence, we can define recursively the sequence of random variable L^p-integrable $(V_k)_{0 \leqslant k \leqslant n}$

$$\begin{cases} V_n = e^{-rt_n}\, \psi_n(X_n, Y_n), \\ V_k = \max\left(e^{-rt_k}\, \psi_k(X_k, Y_k), \mathbb{E}[V_{k+1} \mid \mathcal{F}_k] \right), \qquad 0 \leqslant k \leqslant n-1 \end{cases}$$

called *Backward Dynamical Programming Principle*.

Bermudan Options

Using the Product Recursive Quantizer

We approximate the *Backward Dynamical Programming Principle* by the following sequence involving the couple $(\widehat{X}_k, \widehat{Y}_k)_{0 \leqslant k \leqslant n}$

$$\begin{cases} \widehat{V}_n = \mathrm{e}^{-rt_n} \psi_n(\widehat{X}_n, \widehat{Y}_n), \\ \widehat{V}_k = \max\Big(\mathrm{e}^{-rt_k} \psi_k(\widehat{X}_k, \widehat{Y}_k), \mathbb{E}\big[\widehat{V}_{k+1} \mid (\widehat{X}_k, \widehat{Y}_k) \big] \Big), \qquad k = 1, \dots, n-1. \end{cases}$$

Bermudan Options

Using the Product Recursive Quantizer

The last equation can be rewritten

$$\begin{cases} \widehat{v}_n(x_{i_1}^n, y_{i_2}^n) = e^{-rt_n}\,\psi_n(x_{i_1}^n, y_{i_2}^n), \\ \widehat{v}_k(x_{i_1}^k, y_{i_2}^k) = \max\left(e^{-rt_k}\,\psi_k(x_{i_1}^k, y_{i_2}^k), \sum_{j_1=1}^{N_1}\sum_{j_2=1}^{N_2} \pi_{(i_1,i_2),(j_1,j_2)}^k\, \widehat{v}_{k+1}(x_{j_1}^{k+1}, y_{j_2}^{k+1}) \right), \end{cases}$$

with $\pi_{(i_1,i_2),(j_1,j_2)}^k = \mathbb{P}\left(\widehat{X}_{k+1} = x_{j_1}^{k+1}, \widehat{Y}_{k+1} = y_{j_2}^{k+1} \mid \widehat{X}_k = x_{i_1}^k, \widehat{Y}_k = y_{i_2}^k\right)$.

Finally, the approximation of the price of the bermudan option is given by

$$\mathbb{E}\left[\widehat{v}_k(x_0, \widehat{Y}_0)\right] = \sum_{i=1}^{N_2} p_i\, \widehat{v}_k(x_0, y_i^0)$$

with $p_i = \mathbb{P}(\widehat{Y}_0 = y_i^0)$.

Bermudan Options - Numerical examples ($T = 0.5$ - Call/Put with $K = 100$)

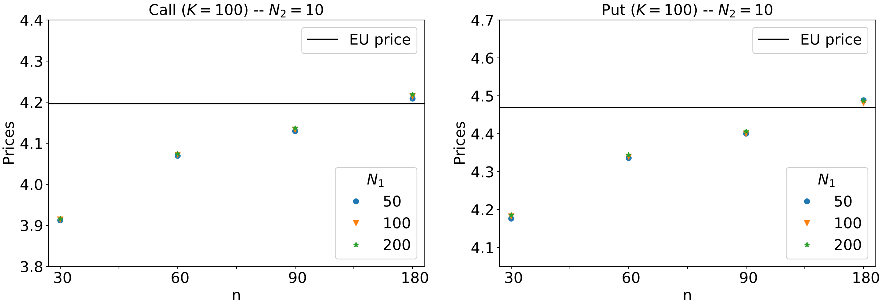

Figure: *Prices of Bermudan options in the stationary Heston model given by product hybrid recursive quantization with fixed value* $N_2 = 10$.

Barrier Options

A Barrier option is a path-dependent financial product whose payoff at maturity date T depends on the value of the process X_T at date T and its maximum or minimum over the period $[0, T]$.
More precisely, we are interested by options with the following types of payoff h

$$h = f(X_T)\,\mathbb{1}_{\{\sup_{t\in[0,T]} X_t \in I\}} \qquad or \qquad h = f(X_T)\,\mathbb{1}_{\{\inf_{t\in[0,T]} X_t \in I\}}$$

where I is an unbounded interval of $\mathbb{R}$, T is the maturity date and f can be any vanilla payoff function (Call, Put, Spread, Butterfly, ...).

Barrier Options

Using a representation formula

Now, using the representation formula based on the **conditional law of the Brownian Bridge** for the price of up-and-out options $\bar{P}_{UO}$ and down-and-out options $\bar{P}_{DO}$

$$\bar{P}_{UO} = e^{-rT}\,\mathbb{E}\left[f(\bar{X}_T)\,\mathbb{1}_{\sup_{t\in[0,T]}\bar{X}_t\leqslant L}\right] = e^{-rT}\,\mathbb{E}\left[f(\bar{X}_T)\prod_{k=0}^{n-1} G^k_{(\overline{X}_k,\overline{Y}_k),\bar{X}_{k+1}}(L)\right]$$

where L is the barrier and

$$G^k_{(x,y),z}(u) = \left(1 - e^{-2n\frac{(x-u)(z-u)}{T\sigma^2(t_k,x,y)}}\right)\mathbb{1}_{\{u\geqslant \max(x,z)\}}\,.$$

Equivalent formulas for other standard Barrier options.

Barrier Options

Using the Product Recursive Quantizer

Finally, replacing $(\bar{X}_k, \bar{Y}_k)$ by $(\widehat{X}_k, \widehat{Y}_k)$ and using a recursive algorithm yield

$$\begin{cases} \widehat{V}_n = \mathrm{e}^{-rT} f(\widehat{X}_n), \\ \widehat{V}_k = \mathbb{E}\left[g_k(\widehat{X}_k, \widehat{Y}_k, \widehat{X}_{k+1}) \widehat{V}_{k+1} \mid (\widehat{X}_k, \widehat{Y}_k) \right], \qquad 0 \leqslant k \leqslant n-1 \end{cases}$$

that can be rewritten

$$\begin{cases} \widehat{v}_n(x_{i_1}^n, y_{i_2}^n) = \mathrm{e}^{-rT} f(x_{i_1}^n), \\ \widehat{v}_k(x_{i_1}^k, y_{i_2}^n) = \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} \pi_{(i_1,i_2),(j_1,j_2)}^k \widehat{v}_{k+1}(x_{j_1}^{k+1}, y_{j_2}^{k+1}) g_k(x_{i_1}^k, y_{i_2}^k, x_{j_1}^{k+1}), \end{cases}$$

with $\pi_{(i_1,i_2),(j_1,j_2)}^k = \mathbb{P}\left(\widehat{X}_{k+1} = x_{j_1}^{k+1}, \widehat{Y}_{k+1} = y_{j_2}^{k+1} \mid \widehat{X}_k = x_{i_1}^k, \widehat{Y}_k = y_{i_2}^k \right)$ and $g_k(x,y,z) = G_{(x,y),z}^k(L)$.

Barrier Options - Numerical example ($T = 0.5$ - Call with $K = 100$ and $L = 115$)

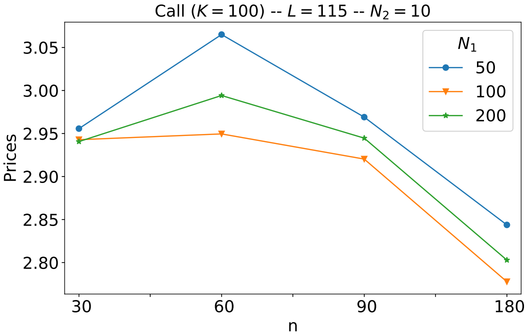

Figure: *Prices of Barrier options with strike $K = 100$ in the stationary Heston model given by product hybrid recursive quantization with fixed value $N_2 = 10$.*

Conclusion

So far

- Introduced a model with steeper smile volatility surface for short maturities than Standard Heston model.
- Fast numerical solution for the pricing of European, Bermudan and Barrier options.
- Calibration.

And more..

- Asian Options

Thank you for your attention!