Stationary Heston model: Calibration and Pricing of exotics using Optimal Quantization

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Stationary Heston model

The model

Dynamic of the asset price process $(S_t^{v_0})_{t\geqslant 0}$ and its volatility $(v_t)_{t\geqslant 0}$ is given by

$$\begin{cases} dS_t^{v_0} = S_t^{v_0}\left((r-q)dt + \sqrt{v_t}dW_t\right) \\ dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t}d\widetilde{W}_t \end{cases}$$

- $S_0 = s_0$ is the initial value of the process, r the spot rate, q the dividend rate,
- κ the mean reverting term,
- θ the long run average price variance,
- ξ is the volatility of the volatility,
- $(W, \widetilde{W})$ is a standard correlated 2d Brownian motion with correlation ρ,

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- $(W, \widetilde{W})$ is a standard correlated 2d Brownian motion with correlation ρ,
- **$v_0 \sim \Gamma(\alpha, \beta)$ with $\beta = (2\kappa)/\xi^2$ and $\alpha = \theta\beta$.**

Remark: 4 parameters $\Longrightarrow$ 1 less than the Standard Heston Model.

History

Introduced by Pagès et Panloup in 2009 then studied by Jacquier and Shi in 2017.

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Generic expression

The price of the European option on the asset $S_T^{v_0}$ is given by

$$I_0 = \mathbb{E}\left[e^{-rT} \varphi(S_T^{v_0}) \right].$$

After preconditioning by v_0, we have

$$I_0 = \mathbb{E}\Big[\mathbb{E}\left[e^{-rT} \varphi(S_T^{v_0}) \mid \sigma(v_0) \right] \Big] = \mathbb{E}\left[f(v_0) \right]$$

where $f(v)$ is the price of the European option in the Standard Heston model with deterministic initial conditions for a given initial volatility v.

Example - Call option

If φ is the payoff of a Call option then f is simply the price given by Fourier transform in the standard Heston model of the European Call Option. Then

$$I_0 = \mathbb{E}\left[\mathrm{e}^{-rT}(S_T^{v_0} - K)_+\right] = \mathbb{E}\left[C\big(\phi(v_0), K, T\big)\right]$$

with

$$C(\phi(v), K, T) = S_0\,\mathrm{e}^{-qT}\,P_1\big(\phi(v), K, T\big) - K\,\mathrm{e}^{-rT}\,P_2\big(\phi(v), K, T\big)$$

and

$$P_1\big(\phi(v), K, T\big) = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \mathrm{Re}\left(\frac{\mathrm{e}^{-iu\log(K)}}{iu}\,\frac{\psi\big(\phi(v), u-i, T\big)}{S_0\,\mathrm{e}^{(r-q)T}}\right)du$$

$$P_2\big(\phi(v), K, T\big) = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \mathrm{Re}\left(\frac{\mathrm{e}^{-iu\log(K)}}{iu}\psi\big(\phi(v), u, T\big)\right)du$$

where $\phi(v) = (S_0, r, q, \rho, \theta, \kappa, \xi, v)$.

Practical aspects

Fixed-point quadratures

- I_0 can be written as an integral against the Laguerre weighting function

$$I_0 = \int_0^{+\infty} f(v) \frac{\beta^\alpha}{\Gamma(\alpha)} v^{\alpha-1} \mathrm{e}^{-\beta v} \, dv = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} f(v) \omega(v) dv$$

where $\omega(v) = v^{\alpha-1} \mathrm{e}^{-\beta v}$ is the Laguerre weighting function.

- Then, for a fixed $n > 0$ with ω_i's and v_i's the associated Laguerre weights and nodes, I_0 would be approximated by

$$\widetilde{I}_0^n = \frac{\beta^\alpha}{\Gamma(\alpha)} \sum_{i=1}^{n} \omega_i f(v_i).$$

Quantization-based cubature method

Approximate I_0 using the following quantization-based cubature formula

$$\widehat{I}_0^N = \mathbb{E}\left[f(\widehat{v}_0^N) \right] = \sum_{i=1}^{N} f(v_{0,i}^N) \, \mathbb{P}(\widehat{v}_0^N = v_{0,i}^N).$$

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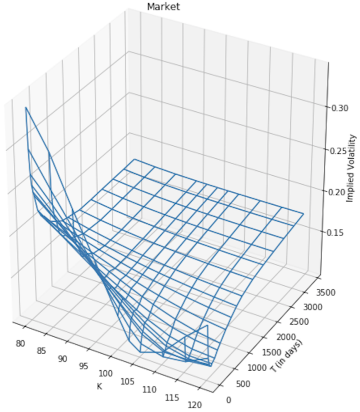
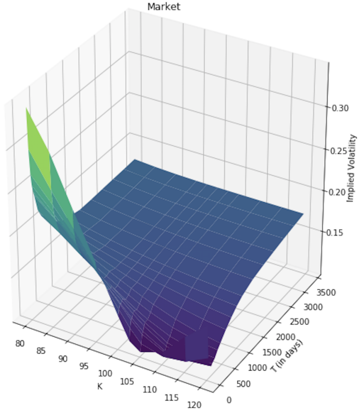
The implied volatility surface from the market

Euro Stoxx 50 - 26th September 2019: $S_0 = 3541$, $r = -0.32\%$, $q = 0.225\%$





The problem

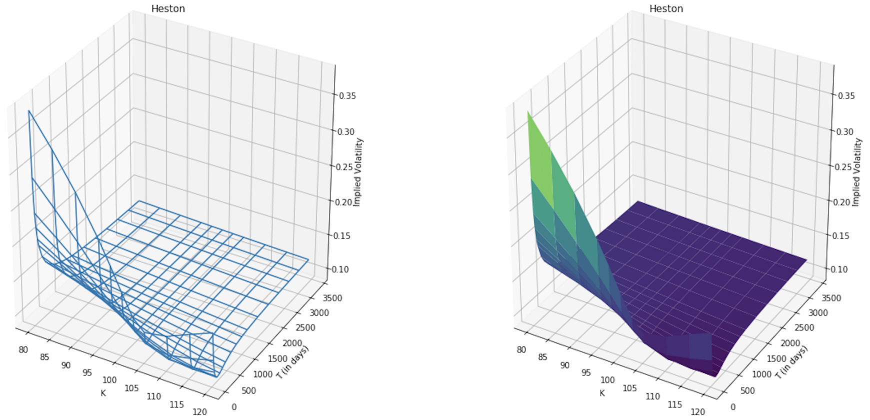
We search for the set of parameters $\phi^\star$ $((\rho^\star, \theta^\star, \kappa^\star, \xi^\star)$ or $(\bar{\rho}^\star, \bar{\theta}^\star, \bar{\kappa}^\star, \bar{\xi}^\star, \bar{v}^\star))$ solution to the penalized and weighted following minimization problem

$$\min_{\phi=(\rho,\theta,\kappa,\xi)} \sum_{K,T} \omega_K \left(\frac{\sigma_{\text{iv}}^{Market}(K,T) - \sigma_{\text{iv}}^{Model}(\phi,K,T)}{\sigma_{\text{iv}}^{Market}(K,T)} \right)^2 + \lambda(\theta + \kappa + \xi)$$

where

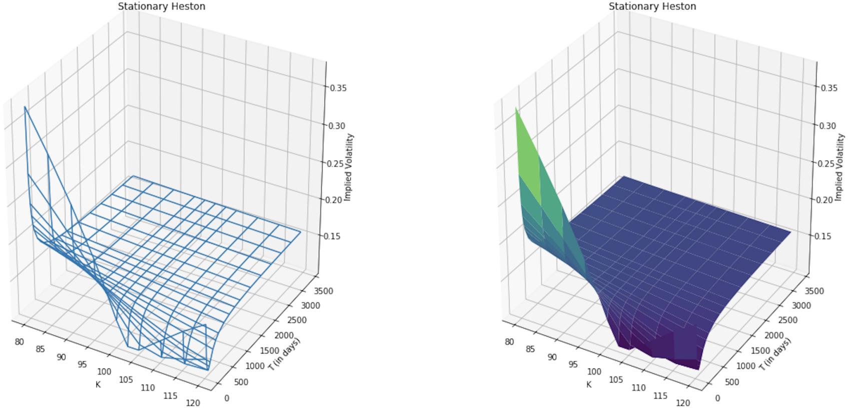
- $\sigma_{\text{iv}}^{Market}(K,T)$ is the implied volatility deduced from the market,
- $\sigma_{\text{iv}}^{Model}(\phi,K,T)$ is the implied volatility of the EU Call price computed for a model: Stationary Heston with parameters $(\rho, \theta, \kappa, \xi)$ or Standard Heston with parameters $(\bar{\rho}, \bar{\theta}, \bar{\kappa}, \bar{\xi}, \bar{v})$,
- ω_K are weights chosen before the optimization in order to give a greater focus on some strikes,
- λ is a penalization factor ($\lambda = 0.00001$).

After Calibration - Standard Heston surface



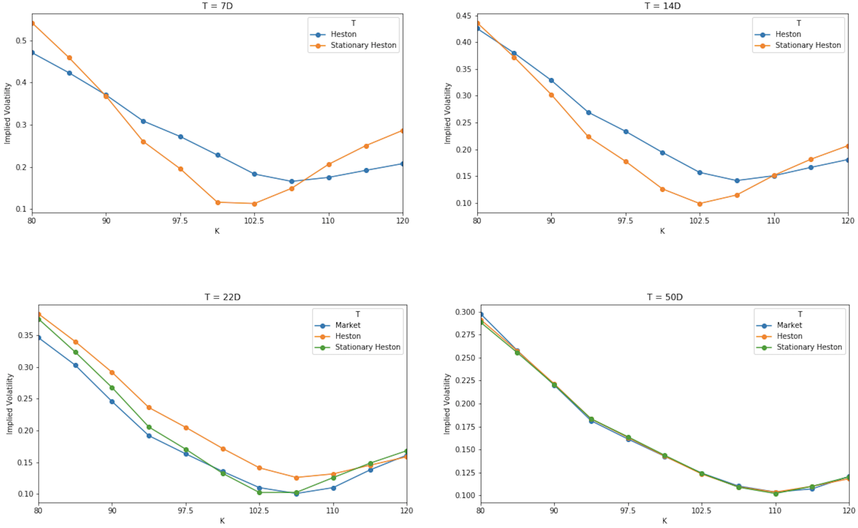
$\bar{\rho} = -0.756$, $\bar{v} = 0.079$, $\bar{\theta} = 0.016$, $\bar{\kappa} = 39.06$ and $\bar{\sigma} = 2.69$

After Calibration - Stationary Heston surface



$\rho = -0.789$, $\theta = 0.028$, $\kappa = 135.72$ and $\sigma = 8.07$

Calibration to expiry 50 days ($T = 50/365$). Relative calibration errors: $< 3\%$ for each implied volatility.



The standard Heston model fails to produce the desired smile for very small maturities while the Stationary model has no problem to generate it with 1 parameter less.

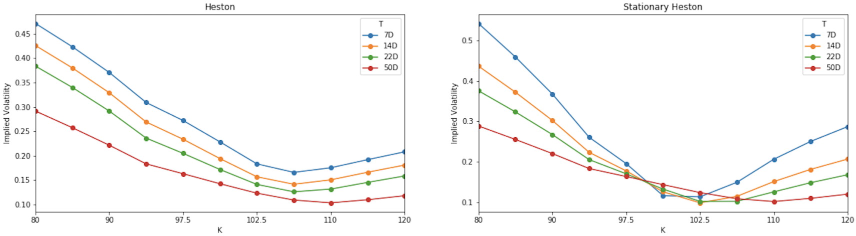


Figure: Term-structure of the volatility in function of T and K of both models (left: Standard Heston and right: Stationary Heston) after calibration at 50 days.

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Definitions

Let $\Gamma_N = \{x_1^N, \dots, x_N^N\} \subset \mathbb{R}$, a subset of size N, called N-quantizer, we define

- The *Voronoï partition* of $\mathbb{R}$ induced by the N-quantizer

$$C_i(\Gamma_N) = \big(x_{i-1/2}^N, x_{i+1/2}^N\big], \quad i \in [\![1, N-1]\!], \quad C_N(\Gamma_N) = \big(x_{N-1/2}^N, x_{N+1/2}^N\big).$$

  Easily defined in dimension one.

- The *Voronoï Quantization* of the random variable X

$$\widehat{X}^{\Gamma_N} = \mathrm{Proj}_{\Gamma_N}(X) = \sum_{i=1}^{N} x_i^N \, \mathbb{1}_{X \in C_i(\Gamma_N)}$$

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- It is convenient to define the quadratic distortion function at level N

$$\mathcal{Q}_{2,\mathrm{N}} : x = (x_1^N, \dots, x_N^N) \longmapsto \mathbb{E}\Big[\min_{i \in [\![1,N]\!]} |X - x_i^N|^2\Big] = \|X - \widehat{X}^N\|_2^2.$$

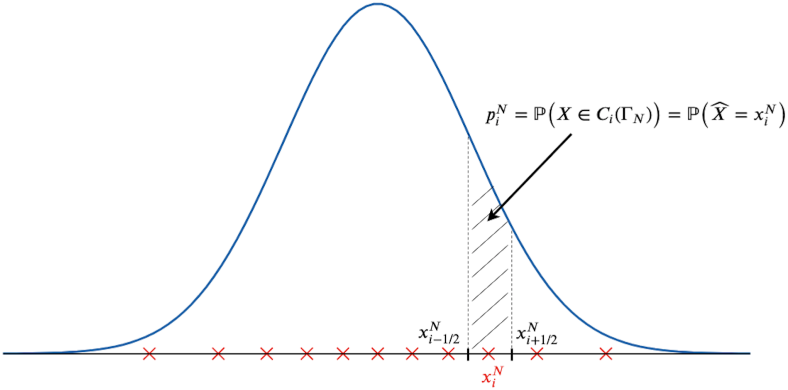

Figure: Gaussian Optimal Quantization

How to build an Optimal Quantizer?

1. Differentiate the $\mathcal{Q}_{2,\mathrm{N}}$

The gradient is given by

$$\nabla\, \mathcal{Q}_{2,\mathrm{N}}(x_{\mathbf{1}:N}) = \left(\mathbb{E}\left[(x_i^N - X)\, \mathbb{1}_{X \in \left(x^N_{i-\mathbf{1/2}}, x^N_{i+\mathbf{1/2}}\right]} \right] \right)_{i=1,\dots,N}$$

2. Solve the fixed point problem

Find $x_{1:N}$ that cancel the gradient

$$\nabla\, \mathcal{Q}_{2,\mathrm{N}}\left(x_{\mathbf{1}:N}\right) = 0 \iff x_i^N = \frac{\mathbb{E}\left[X\, \mathbb{1}_{X \in \left(x^N_{i-\mathbf{1/2}}, x^N_{i+\mathbf{1/2}}\right]} \right]}{\mathbb{P}\left(X \in \left(x^N_{i-1/2}, x^N_{i+1/2}\right] \right)}, \qquad i = 1, \dots, N$$

$$\iff x_i^N = \frac{K_x\left(x^N_{i+1/2}\right) - K_x\left(x^N_{i-1/2}\right)}{F_x\left(x^N_{i+1/2}\right) - F_x\left(x^N_{i-1/2}\right)}, \qquad i = 1, \dots, N.$$

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Litterature

Recursive Quantization

- *Recursive marginal quantization of the Euler scheme of a diffusion process* by G. Pagès and A. Sagna. (2015)
- *Recursive Marginal Quantization of Higher-Order Schemes* by J. Kienitz, T. A. McWalter, E. Platen and R. Rudd. (2017)
- *Product Markovian quantization of an R^d-valued Euler scheme of a diffusion process with applications to finance* by L. Fiorin, G. Pagès and A. Sagna. (2018)

Previous work on Heston model using Quantization

- *Pricing via Quantization in Stochastic Volatility Models* by G. Callegaro, L. Fiorin and M. Grasselli. (2016)
- *Fast Quantization of Stochastic Volatility Models* by J. Kienitz, T. A. McWalter, E. Platen and R. Rudd. (2017)
- *American quantized calibration in stochastic volatility* by G. Callegaro, L. Fiorin and M. Grasselli. (2018)
- And more...

Model transformation

$$\begin{cases} dS_t^{v_0} = S_t^{v_0}\left(rdt + \sqrt{v_t}dW_t\right) \\ dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t}d\widetilde{W}_t \end{cases}$$

Remark

v_t is autonomous, hence $1d$ problem.

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Remark

v_t is autonomous, hence $1d$ problem.

We will be working with (X_t, Y_t) that are transformation of $(S_t^{v_0}, v_t)$

- For the volatility $\longrightarrow Y_t = \mathrm{e}^{\kappa t}\, v_t$.
- For the asset $\longrightarrow X_t = \log(S_t^{v_0})$.

First, the volatility

Milstein Scheme (preserving the positivity)

We consider the following *boosted* volatility process: let $Y_t = e^{\kappa t} v_t$, $t \in [0, T]$.

$$dY_t = e^{\kappa t} \kappa\theta dt + \xi e^{\frac{\kappa t}{2}} \sqrt{Y_t} d\widetilde{W}_t.$$

Now, if we look at the Milstein discretization scheme of Y_t

$$\bar{Y}_{t_{k+1}} = \mathcal{M}^{\Delta}_{\widetilde{b},\widetilde{\sigma}}\big(t_k, \bar{Y}_{t_k}, \widetilde{Z}_{k+1}\big)$$

where $\widetilde{Z}_{k+1} \sim \mathcal{N}(0,1)$ and

$$\mathcal{M}^{\Delta}_{\widetilde{b},\widetilde{\sigma}}(t,x,z) = x - \frac{\widetilde{\sigma}(t,x)}{2\widetilde{\sigma}'_x(t,x)} + \Delta\left(\widetilde{b}(t,x) - \frac{\widetilde{\sigma}\widetilde{\sigma}'_x(t,x)}{2}\right) + \frac{\widetilde{\sigma}\widetilde{\sigma}'_x(t,x)\Delta}{2}\left(z + \frac{1}{\sqrt{\Delta}\widetilde{\sigma}'_x(t,x)}\right)^2$$

with

$$\widetilde{b}(t,x) = e^{\kappa t} \kappa\theta, \qquad \widetilde{\sigma}(t,x) = \xi\sqrt{x}\, e^{\frac{\kappa t}{2}} \quad \text{and} \quad \widetilde{\sigma}'_x(t,x) = \frac{\xi e^{\frac{\kappa t}{2}}}{2\sqrt{x}}.$$

Then, the log-asset

Euler-Maruyama scheme

We consider the logarithm of the asset $X_t = \log(S_t^{v_0})$, yielding

$$dX_t = \left(r - \frac{v_t}{2}\right) dt + \sqrt{v_t} dW_t.$$

Now, using an Euler-Maruyama scheme for the discretization of X_t, we have

$$\begin{cases} \bar{X}_{t_{k+1}} = \mathcal{E}_{b,\sigma}^{\Delta}(t_k, \bar{X}_{t_k}, \bar{Y}_{t_k}, Z_{k+1}) \\ \bar{Y}_{t_{k+1}} = \mathcal{M}_{\widetilde{b},\widetilde{\sigma}}^{\Delta}(t_k, \bar{Y}_{t_k}, \widetilde{Z}_{k+1}) \end{cases}$$

where $Z_{k+1} \sim \mathcal{N}(0,1)$ and $\mathbb{C}\mathrm{orr}(Z_{k+1}, \widetilde{Z}_{k+1}) = \rho$ and

$$\mathcal{E}_{b,\sigma}^{\Delta}(t,x,y,z) = x + b(t,x,y)\Delta + \sigma(t,x,y)\sqrt{\Delta}\, z$$

with

$$b(t,x,y) = r - \frac{e^{-\kappa t}\, y}{2} \qquad \text{and} \qquad \sigma(t,x,y) = \sqrt{e^{-\kappa t}\, y}.$$

First, the volatility

We build recursively the Markovian quantization tree $(\widehat{Y}_k)_{k\in[\![0,n]\!]}$ where $\widehat{Y}_{t_{k+1}}$ is the Voronoï quantization of $\widetilde{Y}_{k+1}$ defined by

$$\widetilde{Y}_{k+1} = \mathcal{M}^{\Delta}_{\widetilde{b},\widetilde{\sigma}}\big(t_k, \widehat{Y}_k, \widetilde{Z}_{k+1}\big), \qquad \widehat{Y}_{t_{k+1}} = \mathrm{Proj}_{\Gamma^{Y}_{N_2}}\big(\widetilde{Y}_{k+1}\big)$$

with $\Gamma^{Y}_{N_2} = \{y_1^{k+1}, \dots, y_{N_2}^{k+1}\}$ the optimal N_2-quantizer of $\widetilde{Y}_{k+1}$ and $\widetilde{Z}_{k+1} \sim \mathcal{N}(0,1)$.

Then, the log-asset

Now, using the fact that $(Y_t)_t$ has already been quantized and the Euler-Maruyama scheme of $(X_t)_t$, we define the Markov quantized scheme

$$\widetilde{X}_{k+1} = \mathcal{E}^{\Delta}_{b,\sigma}\big(t_k, \widehat{X}_k, \widehat{Y}_k, Z_{k+1}\big), \qquad \widehat{X}_{k+1} = \mathrm{Proj}_{\Gamma^X_{N_1}}\left(\widetilde{X}_{k+1}\right)$$

with $\Gamma^X_{N_1} = \{x^k_1, \dots, x^k_{N_1}\}$ the optimal N_1-quantizer of $\widetilde{X}_{k+1}$ and $Z_{k+1} \sim \mathcal{N}(0,1)$.

Bermudan Options

Its price, at time t_0, is given by

$$V_0 = \sup_{\tau \in \{t_1, \cdots, t_n\}} \mathbb{E}\left[e^{-r\tau} \psi_\tau(X_\tau, Y_\tau) \mid \mathcal{F}_{t_0} \right].$$

Hence, we can define recursively the sequence of random variable L^p-integrable $(V_k)_{0 \leqslant k \leqslant n}$

$$\begin{cases} V_n = e^{-rt_n} \psi_n(X_n, Y_n), \\ V_k = \max\left(e^{-rt_k} \psi_k(X_k, Y_k), \mathbb{E}[V_{k+1} \mid \mathcal{F}_k] \right), \qquad 0 \leqslant k \leqslant n-1 \end{cases}$$

called *Backward Dynamical Programming Principle*.

Bermudan Options

Using the Product Recursive Quantizer

We approximate the *Backward Dynamical Programming Principle* by the following sequence involving the couple $(\widehat{X}_k, \widehat{Y}_k)_{0 \leqslant k \leqslant n}$

$$\begin{cases} \widehat{V}_n = e^{-rt_n} \psi_n(\widehat{X}_n, \widehat{Y}_n), \\ \widehat{V}_k = \max\Big(e^{-rt_k} \psi_k(\widehat{X}_k, \widehat{Y}_k), \mathbb{E}\big[\widehat{V}_{k+1} \mid (\widehat{X}_k, \widehat{Y}_k) \big] \Big), \qquad k = 1, \dots, n-1. \end{cases}$$

Bermudan Options

Using the Product Recursive Quantizer

The last equation can be rewritten

$$\begin{cases} \widehat{v}_n(x^n_{i_1}, y^n_{i_2}) = \mathrm{e}^{-rt_n}\, \psi_n(x^n_{i_1}, y^n_{i_2}), \\ \widehat{v}_k(x^k_{i_1}, y^k_{i_2}) = \max\left(\mathrm{e}^{-rt_k}\, \psi_k(x^k_{i_1}, y^k_{i_2}), \sum_{j_1=1}^{N_1}\sum_{j_2=1}^{N_2} \pi^k_{(i_1,i_2),(j_1,j_2)}\, \widehat{v}_{k+1}(x^{k+1}_{j_1}, y^{k+1}_{j_2})\right), \end{cases}$$

with $\pi^k_{(i_1,i_2),(j_1,j_2)} = \mathbb{P}\left(\widehat{X}_{k+1} = x^{k+1}_{j_1}, \widehat{Y}_{k+1} = y^{k+1}_{j_2} \mid \widehat{X}_k = x^k_{i_1}, \widehat{Y}_k = y^k_{i_2}\right)$.

Finally, the approximation of the price of the bermudan option is given by

$$\mathbb{E}\left[\widehat{v}_k(x_0, \widehat{Y}_0)\right] = \sum_{i=1}^{N_2} p_i\, \widehat{v}_k(x_0, y^0_i)$$

with $p_i = \mathbb{P}(\widehat{Y}_0 = y^0_i)$.

Barrier Options

A Barrier option is a path-dependent financial product whose payoff at maturity date T depends on the value of the process X_T at date T and its maximum or minimum over the period $[0, T]$.
More precisely, we are interested by options with the following types of payoff h

$$h = f(X_T)\,\mathbb{1}_{\{\sup_{t\in[0,T]} X_t \in I\}} \qquad or \qquad h = f(X_T)\,\mathbb{1}_{\{\inf_{t\in[0,T]} X_t \in I\}}$$

where I is an unbounded interval of $\mathbb{R}$, T is the maturity date and f can be any vanilla payoff function (Call, Put, Spread, Butterfly, ...).

Barrier Options

Using a representation formula

Now, using the representation formula based on the **conditional law of the Brownian Bridge** for the price of up-and-out options $\bar{P}_{UO}$ and down-and-out options $\bar{P}_{DO}$

$$\bar{P}_{UO} = e^{-rT}\,\mathbb{E}\left[f(\bar{X}_T)\,\mathbb{1}_{\sup_{t\in[0,T]}\bar{X}_t \leqslant L}\right] = e^{-rT}\,\mathbb{E}\left[f(\bar{X}_T)\prod_{k=0}^{n-1} G^k_{(\bar{X}_k,\bar{Y}_k),\bar{X}_{k+1}}(L)\right]$$

where L is the barrier and

$$G^k_{(x,y),z}(u) = \left(1 - e^{-2n\frac{(x-u)(z-u)}{T\sigma^2(t_k,x,y)}}\right)\mathbb{1}_{\{u\geqslant \max(x,z)\}}\,.$$

Equivalent formulas for other standard Barrier options.

Barrier Options

Using the Product Recursive Quantizer

Finally, replacing $(\bar{X}_k, \bar{Y}_k)$ by $(\widehat{X}_k, \widehat{Y}_k)$ and using a recursive algorithm yield

$$\begin{cases} \widehat{V}_n = \mathrm{e}^{-rT} f(\widehat{X}_n), \\ \widehat{V}_k = \mathbb{E}\left[g_k(\widehat{X}_k, \widehat{Y}_k, \widehat{X}_{k+1}) \widehat{V}_{k+1} \mid (\widehat{X}_k, \widehat{Y}_k) \right], \qquad 0 \leqslant k \leqslant n-1 \end{cases}$$

that can be rewritten

$$\begin{cases} \widehat{v}_n(x_{i_1}^n, y_{i_2}^n) = \mathrm{e}^{-rT} f(x_{i_1}^n), \\ \widehat{v}_k(x_{i_1}^k, y_{i_2}^n) = \displaystyle\sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} \pi_{(i_1,i_2),(j_1,j_2)}^k \widehat{v}_{k+1}(x_{j_1}^{k+1}, y_{j_2}^{k+1}) g_k(x_{i_1}^k, y_{i_2}^k, x_{j_1}^{k+1}), \end{cases}$$

with $\pi_{(i_1,i_2),(j_1,j_2)}^k = \mathbb{P}\left(\widehat{X}_{k+1} = x_{j_1}^{k+1}, \widehat{Y}_{k+1} = y_{j_2}^{k+1} \mid \widehat{X}_k = x_{i_1}^k, \widehat{Y}_k = y_{i_2}^k \right)$ and $g_k(x, y, z) = G_{(x,y),z}^k(L)$.

Conclusion

So far

- Introduced a model with steeper smile volatility surface for short maturities than Standard Heston model.
- Fast numerical solution for the pricing of European, Bermudan and Barrier options.
- Calibration.

And more..

- Asian Options

Thank you for your attention!