New Weak Error bounds and expansions for Optimal Quantization

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10/04/2019

Motivation

- Define a discrete random variable $\widehat{X}^N$ of cardinal N approaching X,
- Allowing us to to approach $\mathbb{E}\, f(X)$ by $\mathbb{E}\, f(\widehat{X}^N)$

$$\mathbb{E}\, f(\widehat{X}^N) = \sum_{i=1}^{N} f(x_i^N)\, \mathbb{P}(\widehat{X}^N = x_i^N),$$

- Then study the error induced by this approximation: finding the highest α such that

$$\lim_{N\to+\infty} N^{\alpha} |\, \mathbb{E}\, f(X) - \mathbb{E}\, f(\widehat{X}^N)| \leq C_{f,X} < +\infty.$$

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Definitions

Let $\Gamma_N = \left\{x_1^N, \ldots, x_N^N\right\} \subset \mathbb{R}$, a subset of size N, called N-quantizer, we define:

- The *Voronoï partition* of $\mathbb{R}$ induced by the N-quantizer:

$$\forall i = \{1, \ldots, N\}, \qquad C_i(\Gamma_N) \subset \left\{\xi \in \mathbb{R}, |\xi - x_i^N| \leq \min_{j \neq i} |\xi - x_j^N| \right\}.$$

Easily defined in dimension one.

- The *Voronoï Quantization* of the random variable X:

$$\widehat{X}^{\Gamma_N} = \mathrm{Proj}_{\Gamma_N}(X) = \sum_{i=1}^{N} x_i^N \, \mathbb{1}_{X \in C_i(\Gamma_N)}$$

Definitions

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- It is convenient to define the quadratic distortion function at level N:

$$\mathcal{Q}_{2,\mathrm{N}} : x = \left(x_1^N, \ldots, x_N^N\right) \longmapsto \mathbb{E}\left(\min_{i \in [\![1,N]\!]} |X - x_i^N|^2\right) = \|X - \widehat{X}^N\|_2^2.$$

Existence

The *optimal L^2-mean quantization problem* consists in minimizing the quadratic distortion function over all grids Γ of size $|\Gamma| \leq N$.

Theorem (Kieffer, Cuesta-Albertos, Pagès, Graf Luschgy)

For every $N \geq 1$, there exists (at least) one quadratic Optimal Quantization grid Γ^N at level N and $N \mapsto \inf_{x \in (\mathbb{R})^N} \mathcal{Q}_{2,\mathrm{N}}(x)$ converges to 0 and is decreasing as long as it is positive.

Definition

A grid associated to any N-tuple solution to the above distortion minimization problem is called an optimal quadratic N-quantizer.

Stationarity

A really interesting and useful property concerning quadratic optimal quantizers is the **stationarity property**.

Proposition (Stationarity)

Assume that the support of $\mathbb{P}_X$ *has at least* N *elements. Any* L^2*-optimal* N*-quantizer* $\Gamma_N \in (\mathbb{R})^N$ *is stationary in the following sense: for every Voronoï quantization* $\widehat{X}^N$ *of* X,

$$\mathbb{E}\left(X \middle| \widehat{X}^N\right) = \widehat{X}^N.$$

Asymptotic behavior in N of $\mathcal{Q}_{2,N}(x)$

Theorem (Zador's Theorem)

Let $X \in L^{2+\delta}_{\mathbb{R}}(\mathbb{P})$ for some $\delta > 0$. Let $\mathbb{P}_x(d\xi) = \varphi(\xi) \cdot \lambda(d\xi) + \nu(d\xi)$, where $\nu \perp \lambda$ i.e. is singular with respect to the Lebesgue measure λ on $\mathbb{R}$. Then, there is a constant $\widetilde{J}_{2,1} \in (0, +\infty)$ such that

$$\lim_{N \to +\infty} N \min_{\Gamma_N \subset \mathbb{R}, |\Gamma_N| \leq N} \|X - \widehat{X}^N\|_2 = \widetilde{J}_{2,1} \left[\int_{\mathbb{R}} \varphi^{\frac{1}{3}} d\lambda \right]^{1+\frac{1}{2}}$$

with $\widetilde{J}_{2,1} = \frac{1}{12}$.

Theorem

Moreover

$$\lim_{N \to +\infty} N^2 \,\mathbb{E}\left[g(\widehat{X}^N) |X - \widehat{X}^N|^2 \right] = \mathcal{Q}_2(\mathbb{P}_x) \int g(\xi) \,\mathbb{P}_x\,(d\xi)$$

for every function $g : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}\, g(X) < +\infty$.

Local behavior of optimal quantizers

Theorem (Local behavior of optimal quantizers)

Let $\mathbb{P}_X$ *be a distribution on the real line with connected support* $\operatorname{supp}(\mathbb{P}_X)$*. Assume that* $\mathbb{P}_X$ *has a probability density function* φ *which is positive and Lipschitz continuous on every compact set of the interior* (m, M) *of* $\operatorname{supp}(\mathbb{P}_X)$*. For every* $[a, b] \subset (m, M)$*,* $a < b$*,*

(a) the weights are asymptotically uniformly distributed

$$\sup_{\left\{i : x_i^N \in [a,b]\right\}} \left| N\,\mathbb{P}_X\left(C_i\left(\Gamma_N\right)\right) - c_{\varphi,r+1}\varphi^{\frac{2}{3}}\left(x_i^N\right)\right| \xrightarrow{N\to+\infty} 0,$$

(b) the local distortion is asymptotically uniformly distributed

$$\sup_{\{i : x_i^N \in [a,b]\}} \left| N^3 \int_{C_i(\Gamma_N)} \left|x_i^N - \xi\right|^2 \mathbb{P}_X(d\xi) - \frac{\|\varphi\|_{1/3}}{12} \right| \xrightarrow{N\to+\infty} 0.$$

L^r-L^s-distortion mismatch

Theorem (L^r-L^s-distortion mismatch)

Let $X : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}$ be a random variable. Assume that the distribution $\mathbb{P}_X$ of X has a non-zero absolutely continuous component with density φ. Let $(\Gamma_N)_{N \geq 1}$ be a sequence of L^2-optimal grids. Let $s \in (2, 3)$. If

$$X \in L^{\frac{s}{3-s}+\delta}(\Omega, \mathcal{A}, \mathbb{P})$$

for some $\delta > 0$, then

$$\limsup_N N \|X - \widehat{X}^N\|_s < +\infty.$$

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What do we mean by weak error bounds for optimal quantization?

Considering X, a random variable in dimension one and the quadratic optimal quantizer at level N, $\widehat{X}^N$ of X, we are interested by the highest α in the following quantity that keeps the limit upper-bounded

$$\lim_{N \to +\infty} N^{\alpha} \big| \mathbb{E} f(X) - \mathbb{E} f(\widehat{X}^N) \big| \leq C_{f,X} < +\infty$$

for different classes of functions.

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Known results

- For **Lipschitz functions**: $\alpha = 1$

$$N|\mathbb{E}\, f(X) - \mathbb{E}\, f(\widehat{X}^N)| \leq [f_{_{Lip}}] N \|X - \widehat{X}^N\|_{_1} \leq N[f_{_{Lip}}] \|X - \widehat{X}^N\|_{_2} \xrightarrow{N \to +\infty} C_f$$

using Zador's Theorem.

- For **differentiable functions with Lipschitz derivative**: $\alpha = 2$ using the following expansion for f

$$f(y) = f(x) + f'(y)(y - x) + \int_0^1 (f'(ty + (1-t)x) - f'(x))(y - x)dt.$$

- For **differentiable functions with α'-Hölder derivative**: $\alpha = 1 + \alpha'$.

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Piecewise affine functions

Lemma

Let $f : \mathbb{R} \to \mathbb{R}$ a piecewise-defined affine function with finitely many breaks of affinity.

(a) *If f is continuous, then there exists a real constant $C_{f,X} > 0$ such that*

$$\limsup_N N^2 \left| \mathbb{E}\, f(X) - \mathbb{E}\, f(\widehat{X}^N) \right| \le C_{f,X} < +\infty. \tag{1}$$

(b) *However, if f is not supposed continuous, then there exists a real constant $C_{f,X} > 0$ such that*

$$\limsup_N N \left| \mathbb{E}\, f(X) - \mathbb{E}\, f(\widehat{X}^N) \right| \le C_{f,X} < +\infty. \tag{2}$$

Lipschitz Convex functions

Representation formula of Lipschitz convex functions

Let I be any interval non trivial ($\neq \emptyset, \{a\}$) with endpoints $a, b \in \overline{\mathbb{R}}$. Then, there exists a unique finite non-negative Borel measure $\nu := \nu_f$ on I such that,

$$f(x) = f(c) + (x - c)f'_+(c) + \int_{[a,c)\cap I} (u - x)_+ \nu(du) + \int_{[c,b]\cap I} (x - u)_+ \nu(du).$$

Proposition

If $\operatorname{supp}(\mathbb{P}_X) \cap \operatorname{supp}(\nu)$ *is compact then there exists a real constant* $C_{f,X} > 0$ *such that*

$$\limsup_N N^2 \left| \mathbb{E}\, f(X) - \mathbb{E}\, f(\widehat{X}^N) \right| \leq C_{f,X} < +\infty.$$

Lipschitz Convex functions

Proof.

$$\mathbb{E}\left[f(X) - f(\widehat{X}^N)\right] = \sum_{i=1}^{N} \mathbb{E}\left[\left(f(X) - f(x_i^N)\right) \mathbb{1}_{\{X \in (x_{i-1/2}^N, x_{i+1/2}^N]\}}\right].$$

Using the representation formula and noticing that

$$\mathbb{E}\left[\left(X - x_i^N\right) f'_+\left(x_i^N\right) \mathbb{1}_{\{X \in C_i(\Gamma_N)\}}\right] = 0,$$

we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left(f(X) - f(x_i^N)\right) \mathbb{1}_{\{X \in (x_{i-1/2}^N, x_{i+1/2}^N]\}}\right] \\
&= \mathbb{E}\left[\int_{(x_{i-1/2}^N, x_i^N)} (u - X)_+ \nu(du) \, \mathbb{1}_{\{X \in (x_{i-1/2}^N, x_i^N]\}}\right] \\
&\qquad + \mathbb{E}\left[\int_{[x_i^N, x_{i+1/2}^N)} (X - u)_+ \nu(du) \, \mathbb{1}_{\{X \in [x_i^N, x_{i+1/2}^N]\}}\right].
\end{aligned}$$

Lipschitz Convex functions

Proof (Cont.)

Now, using a crude upper-bound, we get

$$\mathbb{E}\left[\left(f(X) - f(x_i^N)\right) \mathbb{1}_{\{X \in (x_{i-1/2}^N, x_{i+1/2}^N]\}}\right]$$

$$\leq \mathbb{E}\left[\left(x_i^N - X\right) \nu((x_{i-1/2}^N, x_i^N)) \mathbb{1}_{\{X \in (x_{i-1/2}^N, x_i^N]\}}\right] + \mathbb{E}\left[(X - x_i^N)\nu([x_i^N, x_{i+1/2}^N)) \mathbb{1}_{\{X \in [x_i^N, x_{i+1/2}^N]\}}\right]$$

$$\leq \mathbb{E}\left[|x_i^N - X| \mathbb{1}_{\{X \in C_i(\Gamma_N)\}}\right] \nu(C_i(\Gamma_N))$$

Hence

$$0 \leq \mathbb{E}\left[f(X) - f(\widehat{X}^N)\right] \leq \sum_{i=1}^{N} \mathbb{E}\left[|x_i^N - X| \mathbb{1}_{\{X \in C_i(\Gamma_N)\}}\right] \nu(C_i(\Gamma_N))$$

$$\leq \sum_{i=1}^{N} \mathbb{E}\left[|x_i^N - X| \mathbb{1}_{\{X \in C_i(\Gamma_N)\}}\right] \mathbb{1}_{\{x_i^N \in J_\nu\}} \nu(C_i(\Gamma_N))$$

with $J_\nu := \left[\inf_N x_{i_a - 1/2}^N, \sup_N x_{i_b + 1/2}^N\right]$. Hence

$$N^2 \, \mathbb{E}\left[f(X) - f(\widehat{X}^N)\right] \leq \nu(I_{\mathbb{P}_X}) N^2 \sup_{i: x_i^N \in \mathrm{supp}(\mathbb{P}_X) \cap J_\nu} \mathbb{E}\left[|\widehat{X}^N - X| \mathbb{1}_{\{X \in C_i(\Gamma_N)\}}\right] \xrightarrow{N \to +\infty} C_{f,X} < +\infty.$$

□

Piecewise-defined Differentiable functions

Definitions

A function $f : I \to \mathbb{R}$ is supposed to be **locally-Lipschitz continuous**, if

$$\forall x, y \in I \quad |f(x) - f(y)| \leq [f]_{Lip,loc} |x - y| (c + g(x) + g(y))$$

where $[f]_{Lip,loc}$ is a real constant and $g : \mathbb{R} \to \mathbb{R}_+$.

A function $f : I \to \mathbb{R}$ is supposed to be **locally α-Hölder continuous**, if

$$\forall x, y \in I \quad |f(x) - f(y)| \leq [f]_{\alpha,loc} |x - y|^{\alpha} (c + g(x) + g(y))$$

where $[f]_{\alpha,loc}$ is a real constant and $g : \mathbb{R} \to \mathbb{R}_+$.

Piecewise-defined Differentiable functions

Proposition

If $f : \mathbb{R} \to \mathbb{R}$ is a piecewise-defined continuous function with finitely many breaks of affinity $\{a_1, \dots, a_K\}$, where $-\infty = a_0 < a_1 < \cdots < a_K < a_{K+1} = +\infty$, such that the piecewise-defined derivatives denoted $(f_k')_{k=0,\dots,d}$ are

(a) *locally-Lipschitz continuous on (a_k, a_{k+1}) where $\exists q_k \geq 1$ such that the q_k-th power of $g_k : (a_k, a_{k+1}) \to \mathbb{R}_+$ are convex and $(\|g_k(X)\|_{q_k})_{k=1,K} < +\infty$. Then there exists a real constant $C_{f,X} > 0$ such that*

$$\limsup_N N^2 \left| \mathbb{E}\, f(X) - \mathbb{E}\, f(\widehat{X}^N) \right| \leq C_{f,X} < +\infty.$$

(b) *locally α-Hölder continuous on (a_k, a_{k+1}), $\alpha \in (0,1)$ such that the q_k-th power of $g_k : (a_k, a_{k+1}) \to \mathbb{R}_+$ are convex and $(\|g_k(X)\|_{q_k})_{k=1,K} < +\infty$. Then there exists a real constant $C_{f,X} > 0$ such that*

$$\limsup_N N^{1+\alpha} \left| \mathbb{E}\, f(X) - \mathbb{E}\, f(\widehat{X}^N) \right| \leq C_{f,X} < +\infty.$$

Piecewise-defined Differentiable functions

Ideas in the proof

- Divide the sum of the integral of the difference in two parts: one where the cells contains a break of affinity and the other part where there is not:

$$\mathbb{E}\, f(\widehat{X}^N) - \mathbb{E}\, f(X) = \sum_{i \in I^N_{reg}} \int_{C_i(\Gamma_N)} f(x_i^N) - f(\xi)\, \mathbb{P}_X\,(d\xi) + \sum_{i \notin I^N_{reg}} \int_{C_i(\Gamma_N)} f(x_i^N) - f(\xi)\, \mathbb{P}_X\,(d\xi)$$

- Taking care of the second term in the standard way (Taylor expansion, crude upper-bound, Zador's Theorem and L^r-L^s-distortion mismatch Theorem).
- Now the first term: finite number of terms in the sum, integral representation of f with f' bounded, hence

$$\left| \int_{C_i(\Gamma_N)} f(x_i^N) - f(\xi)\, \mathbb{P}_X\,(d\xi) \right| = \left| \int_{C_i(\Gamma_N)} \int_{\xi}^{x_i^N} f'(u) du\, \mathbb{P}_X\,(d\xi) \right| \leq [f' \,|_{K_0}]_{Lip} \int_{C_i(\Gamma_N)} |\xi - x_i^N|\, \mathbb{P}_X\,(d\xi)\,.$$

Summing among the term and using the Theorem dealing with the local behavior gives use the result.

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Proposition (Weak-Error expansion)

Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function with Lipschitz second derivative. Then, $\forall \beta \in (0,1)$, we have the following expansion

$$\mathbb{E}\, f(X) = \mathbb{E}\, f(\widehat{X}^N) + \frac{c_2}{N^2} + O\left(N^{-(2+\beta)}\right).$$

Moreover, if $\varphi : [a,b] \to \mathbb{R}_+$ is a Lipschitz continuous probability density function, bounded away from 0 on $[a,b]$ then we can choose $\beta = 1$, yielding

$$\mathbb{E}\, f(X) = \mathbb{E}\, f(\widehat{X}^N) + \frac{c_2}{N^2} + O\left(N^{-3}\right).$$

Richardson-Romberg extrapolation

Combine $\widehat{X}^N$ of size N and $\widehat{X}^M$ of size M, with $M > N$ in order to *kill* the residual term, leading

$$\mathbb{E}\, f(X) = \mathbb{E}\left(\frac{M^2 f(\widehat{X}^M) - N^2 f(\widehat{X}^N)}{M^2 - N^2}\right) + O\left(N^{-(2+\beta)}\right). \qquad (3)$$

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Proposition (Weak-Error expansion for product optimal quantizer)

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a twice differentiable function with bounded Hessian. Let $X : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d$ be a random vector with independent components $(X_k)_{k=1,\dots,d}$. For every $(N_k)_{k=1,\dots,d} \geq 1$, let $(\widehat{X}_d^{N_d})_{k=1,\dots,d}$ be quadratic optimal quantizers of $(X_k)_{k=1,\dots,d}$ taking values in the grids $(\Gamma_{N_k})_{k=1,\dots,d}$ respectively and we define $\widehat{X}^N$ as the product quantizer X taking values in the finite grid $\Gamma_N := \bigotimes_{k=1,\dots,d} \Gamma_{N_d}$ of size $N := N_1 \times \cdots \times N_d$. Then, we have the following expansion

$$\mathbb{E}\, f(X) = \mathbb{E}\, f(\widehat{X}^N) + \sum_{k=1}^{d} \frac{c_k}{N_k^2} + O\left(N_1^{-(2+\beta)} \vee \cdots \vee N_d^{-(2+\beta)} \right).$$

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Our quantity of interest

$$I := \mathbb{E}\, f(Z).$$

with $Z \in L^2_{\mathbb{R}^d}(\Omega, \mathcal{A}, \mathbb{P})$ a random vector with components $(Z_k)_{k=1,\ldots,d}$ and $f : \mathbb{R}^d \to \mathbb{R}$ our function of interest.

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d dimensional Quantized Control Variate Ξ_N

$$\Xi^N := \left(\Xi^N_k\right)_{k=1,\ldots,d}$$

where each component Ξ^N_k is defined by

$$\Xi^N_k := f_k(Z_k) - \mathbb{E}\, f_k(\widehat{Z}^N_k),$$

with $f_k(z) := f(\mathbb{E}\, Z_1, \ldots, z, \ldots, \mathbb{E}\, Z_d)$ and $\widehat{Z}^N_k$ is an optimal quantizer of cardinal N of the component Z_k.

Controlled approximation I^N of I

$$I^N = \mathbb{E}\left(f(Z) - \langle \lambda, \Xi^N \rangle\right) = \mathbb{E}\left(f(Z) - \sum_{k=1}^{d} \lambda_k f_k(Z_k)\right) + \sum_{k=1}^{d} \lambda_k \,\mathbb{E}\, f_k(\widehat{Z}_k^N).$$

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Remark (Optimal λ_k's)

We can find easily the λ_k's minimizing the variance of X^λ

$$\mathbb{V}\mathrm{ar}(X^{\lambda_{\min}}) = \min\{\mathbb{V}\mathrm{ar}(f(Z) - \langle \lambda, \Xi^N \rangle), \lambda \in \mathbb{R}^d\}.$$

The solution of the above optimization problem is the solution to the system $D(Z) \cdot \lambda = B$ where $D(Z)$, the covariance-variance matrix of $(f_k(Z_k))_{k=1,\dots,d}$, and B are given by

$$D(Z) = \begin{pmatrix} \mathbb{V}\mathrm{ar}(f_1(Z_1)) & \cdots & \mathbb{C}\mathrm{ov}(f_1(Z_1), f_d(Z_d)) \\ \vdots & \ddots & \vdots \\ \mathbb{C}\mathrm{ov}(f_d(Z_d), f_1(Z_1)) & \cdots & \mathbb{V}\mathrm{ar}(f_d(Z_d)) \end{pmatrix}, \quad B = \begin{pmatrix} \mathbb{C}\mathrm{ov}(f(Z), f_1(Z_1)) \\ \vdots \\ \mathbb{C}\mathrm{ov}(f(Z), f_d(Z_d)) \end{pmatrix}.$$

Monte Carlo estimator of $I^{\lambda,N}$

$$\widehat{I}_M^{\lambda,N} = \frac{1}{M}\sum_{m=1}^{M}\left(f(Z^m) - \sum_{k=1}^{d}\lambda_k f_k(Z_k^m)\right) + \sum_{k=1}^{d}\lambda_k\,\mathbb{E}\,f_k(\widehat{Z}_k^N).$$

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Remark (Biased estimator)

The quantity we are really interested by is not the bias but the *MSE* (Mean Square Error), yielding a *bias-variance decomposition*

$$\mathrm{MSE}(\widehat{I}_M^{\lambda,N}) = \underbrace{\left(\sum_{k=1}^{d}\lambda_k(\mathbb{E}\,f_k(\widehat{Z}_k^N) - \mathbb{E}\,f_k(Z_k))\right)^2}_{bias^2} + \frac{1}{M}\underbrace{\mathbb{V}\mathrm{ar}\left(f(Z) - \sum_{k=1}^{d}\lambda_k f_k(Z_k)\right)}_{Monte\ Carlo\ variance}.$$

Minimizing the cost of the Monte Carlo estimator

Our aim is to minimize the cost of the Monte Carlo simulation for a given *MSE* or upper-bound of the *MSE*.

$$\inf_{MSE(\widehat{I}_M^{\lambda,N})\leq\epsilon^2} \mathrm{Cost}(\widehat{I}_M^{\lambda,N}).$$

Let $\kappa = \mathrm{Cost}(f(z))$ for a given $z \in \mathbb{R}^d$. The global complexity associated to the estimator $\widehat{I}_M^{\lambda,N}$ is given by

$$\mathrm{Cost}(\widehat{I}_M^{\lambda,N}) = \kappa((d+1)M + dN)$$

and if each f_k is in a class of function where the weak error of order two is reached when using a quantization-based cubature formula then our minimization problem becomes

$$\inf_{\frac{C}{N^4}+\frac{\sigma_\lambda^2}{M}\leq\epsilon^2} \kappa((d+1)M + dN).$$

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Vanilla options B&S

Payoff of a Call

$$(S_T - K)_+$$

with price

$$I_0 := \mathbb{E}\left(e^{-rT}(S_T - K)_+\right) = Call_{BS}(S_0, K, r, \sigma, T) = S_0\,\mathcal{N}(d_1) - K\,e^{-rT}\,\mathcal{N}(d_2).$$

Approximation of $\mathbb{E}\left(e^{-rT}(S_T - K)_+\right)$ using Optimal Quantization

- First, we rewrite the expectation in function of Z a normal distributed random variable

  $$\mathbb{E}\left(e^{-rT}(S_T - K)_+\right) = \mathbb{E}\, f(Z)$$

  where $f(x) := e^{-rT}\left(s_0\, e^{(r-\sigma^2/2)T+\sigma\sqrt{T}x} - K\right)_+$.

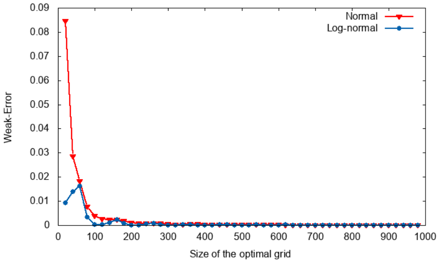
- Second, we have

  $$\mathbb{E}\left(e^{-rT}(S_T - K)_+\right) = \mathbb{E}\, g(S_T)$$

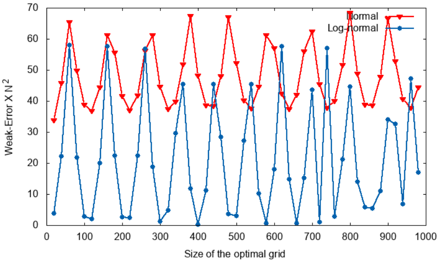
Parameters

$$s_0 = 100, \qquad r = 0.1, \qquad \sigma = 0.5, \qquad T = 1, \qquad K = 80.$$

The reference value is 34.15007.



(a) $N \longmapsto |I_0 - \mathbb{E}\, f(\widehat{Z}^N)|$ (▼) and $N \longmapsto |I_0 - \mathbb{E}\, g(\widehat{X}^N)|$ (●)



(b) $N \longmapsto N^2 \times |I_0 - \mathbb{E}\, f(\widehat{Z}^N)|$ (▼) and $N \longmapsto N^2 \times |I_0 - \mathbb{E}\, g(\widehat{X}^N)|$ (●)

Figure: Call option in a Black-Scholes model.

Compound Option B&S

Payoff of a Put-on-Call

$$\left(K_1 - \mathbb{E}\left[e^{-r(T_2-T_1)}(S_{T_2} - K_2)_+ \mid S_{T_1}\right]\right)_+$$

with price

$$I_0 := \mathbb{E}\left(e^{-rT_1}\left(K_1 - \mathbb{E}\left[e^{-r(T_2-T_1)}(S_{T_2} - K_2)_+ \mid S_{T_1}\right]\right)_+\right)$$
$$= \mathbb{E}\left[e^{-rT_1}\left(K_1 - Call_{BS}(S_{T_1}, K_2, r, \sigma, T_2 - T_1)\right)_+\right]$$

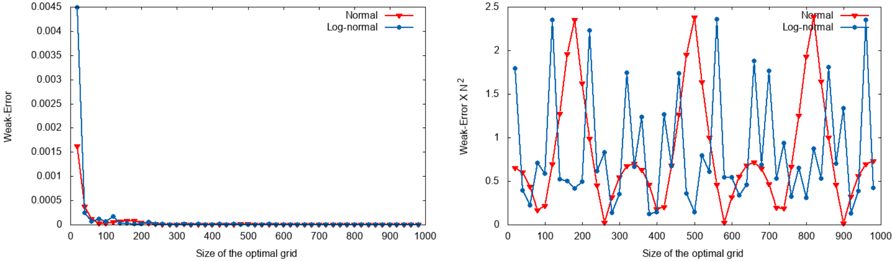
Approximation of I_0 using Optimal Quantization

- First, $I_0 = \mathbb{E}\, f(Z)$ where $Z \sim \mathcal{N}(0;1)$ and $f(Z) = e^{-rT_1}(K_1 - Call_{BS}(s_0\, e^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z}, K_2, r, \sigma, T_2 - T_1))_+$.
- Second, $I_0 = \mathbb{E}\, g(X)$ where $\log(X) \sim \mathcal{N}((r-\sigma^2/2)T; \sigma\sqrt{T})$ and $g(X) = e^{-rT_1}(K_1 - Call_{BS}(s_0 X, K_2, r, \sigma, T_2 - T_1))_+$.

Parameters

$$s_0 = 100, r = 0.03, \sigma = 0.2,\ T_1 = \frac{1}{12},\ T_2 = \frac{1}{2}, K_1 = 6.5, K_2 = 100.$$

The reference value is 1.3945704.



(a) $N \longmapsto |I_0 - \mathbb{E}\, f(\widehat{Z}^N)|$ (▼) and $N \longmapsto |I_0 - \mathbb{E}\, g(\widehat{X}^N)|$ (●)

(b) $N \longmapsto N^2 \times |I_0 - \mathbb{E}\, f(\widehat{Z}^N)|$ (▼) and $N \longmapsto N^2 \times |I_0 - \mathbb{E}\, g(\widehat{X}^N)|$ (●)

Figure: Put-On-Call option in a Black-Scholes model.

Exchange spread Option B&S

Exchange spread Option

$$\left(S_T^1 - S_T^2 - K\right)_+$$

with price

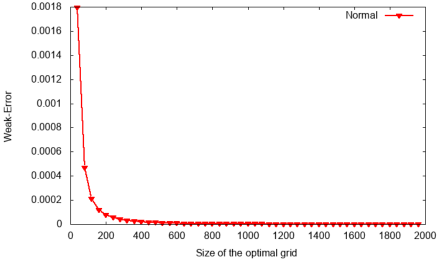
$$\begin{aligned} I_0 &:= \mathbb{E}\left(e^{-rT}\left(S_T^1 - S_T^2 - K\right)_+\right) \\ &= \mathbb{E}\left[Call_{BS}\left(s_0^1\, e^{-\rho^2\sigma_1^2 T/2+\sigma_1\rho\sqrt{T}Z_2}, s_0^2\, e^{(r-\sigma_2^2/2)T+\sigma_2\sqrt{T}Z_2} + K, r, \sigma_1\sqrt{1-\rho^2}, T\right)\right] \end{aligned}$$

where $Z_2 \sim \mathcal{N}(0,1)$.

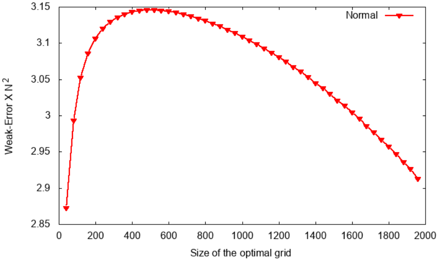
Parameters

$$s_0^i = 100, \quad r = 0.02, \quad \sigma_i = 0.5, \quad \rho = 0.5, \quad T = 10, \quad K = 10.$$

The reference value is 53.552678.



(a) $N \longmapsto |I_0 - \mathbb{E}\, g(\widehat{Z}^N)|$ (▼)



(b) $N \longmapsto N^2 \times |I_0 - \mathbb{E}\, g(\widehat{Z}^N)|$ (▼)

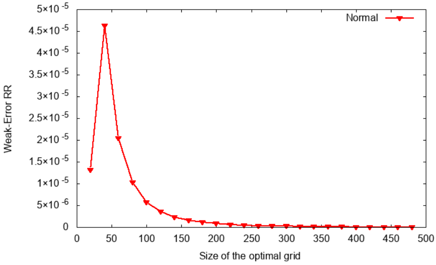
Figure: Exchange spread option pricing in a Black-Scholes model.

Remark

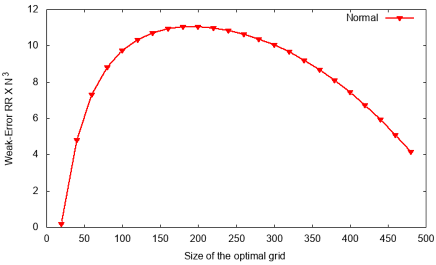
Noticing that $g(z)$ is a twice differentiable function with bounded second derivative, we can reach a weak error of order 3 when using a Richardson-Romberg extrapolation denoted $\widehat{I}^{RR}_{M,N}$ and defined by

$$\widehat{I}^{RR}_{M,N} := \mathbb{E}\left(\frac{M^2 g(\widehat{Z}^M) - N^2 g(\widehat{Z}^N)}{M^2 - N^2}\right).$$

For the next figure, we chose $M := k \times N$ with $k = 1.2$.



(a) $N \longmapsto |I_0 - \widehat{I}^{RR}_{M,N}|$ (▼)



(b) $N \longmapsto N^3 \times |I_0 - \widehat{I}^{RR}_{M,N}|$ (▼)

Figure: Richardson-Romberg extrapolation, for Exchange spread option in a B&S model.

Basket Option B&S

Payoff of a Basket

$$f(S_t^1, \dots, S_T^d) := \left(\sum_{k=1}^{d} \alpha_k S_T^k - K\right)_+ \quad \text{with price } I_0 := \mathrm{e}^{-rT}\,\mathbb{E}\left(\sum_{k=1}^{d} \alpha_k S_T^k - K\right)_+ .$$

Approximation of I_0 using a Monte Carlo estimator

- First, the Crude Monte Carlo estimator

$$\widehat{I}_M := \mathrm{e}^{-rT}\,\frac{1}{M}\sum_{m=1}^{M}\left(\sum_{k=1}^{d} \alpha_k S_T^{k,(m)} - K\right)_+ .$$

- Second, the controlled Monte Carlo estimator

$$\widehat{I}_M^{\lambda,N} := \mathrm{e}^{-rT}\,\frac{1}{M}\sum_{m=1}^{M}\left(\sum_{k=1}^{d} \alpha_k S_T^{k,(m)} - K\right)_+ - \langle \lambda, \Xi^{N,(m)} \rangle.$$

Two possible control variates

- First, we consider

$$\Xi_k^N := f(\mathbb{E}\, S_T^1, \cdots, S_T^k, \cdots, \mathbb{E}\, S_T^d) - \mathbb{E}\, f(\mathbb{E}\, S_T^1, \cdots, \widehat{S}_T^{k,N}, \cdots, \mathbb{E}\, S_T^d).$$

  In that case, the Monte Carlo estimator is denoted $\widehat{\overline{I}}_M^{\lambda,N}$.

- Second, we consider

$$\widetilde{\Xi}_k^N := g(0, \cdots, Z^k, \cdots, 0) - \mathbb{E}\, g(0, \cdots, \widehat{Z}^N, \cdots, 0)$$

  where $(Z^k)_{k=1,\ldots,d}$ are i.i.d Gaussian random variables and $(\widehat{Z}^N)_{k=1,\ldots,d}$ is an optimal quantizer of $Z \sim \mathcal{N}(0,1)$. In that case, the Monte Carlo estimator is denoted $\widehat{\widetilde{\overline{I}}}_M^{\lambda,N}$.

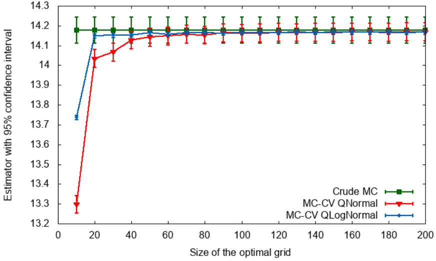
Parameters

$$s_0^i = 100, \qquad r = 2\%, \qquad \sigma_i = i/(d+1), \quad \rho = 0.5$$

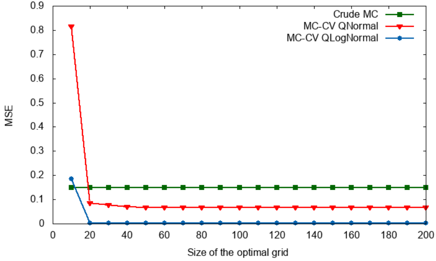
and the specifications of the product are

$$K = 100, \quad \alpha_i = 2i/(d(d+1))$$

		$N = 20$		$N = 200$	
d	MC Estimator	Mean ($\pm$1.96$\times$std)	MSE	Mean ($\pm$1.96$\times$std)	MSE
$d = 2$	Crude	14.2695 ($\pm$0.0662)	0.1450	14.2695 ($\pm$0.0662)	0.1450
	CV Gaussian	14.1017 ($\pm$0.0399)	0.0774	14.2773 ($\pm$0.0399)	0.0530
	CV Log-Normal	14.2351 ($\pm$0.0078)	0.0026	14.2614 ($\pm$0.0078)	0.0020
$d = 3$	Crude MC	14.1770 ($\pm$0.0671)	0.1492	14.1770 ($\pm$0.0671)	0.1492
	CV Gaussian	14.0336 ($\pm$0.0451)	0.0837	14.1685 ($\pm$0.0451)	0.0673
	CV Log-Normal	14.1479 ($\pm$0.0104)	0.0038	14.1674 ($\pm$0.0104)	0.0036
$d = 5$	Crude MC	13.8803 ($\pm$0.0720)	0.1717	13.8803 ($\pm$0.0720)	0.1717
	CV Gaussian	13.6686 ($\pm$0.0562)	0.1580	13.8883 ($\pm$0.0562)	0.1044
	CV Log-Normal	13.8797 ($\pm$0.0151)	0.0080	13.9008 ($\pm$0.0151)	0.0076
$d = 10$	Crude MC	13.5046 ($\pm$0.0599)	0.1186	13.5046 ($\pm$0.0599)	0.1186
	CV Gaussian	13.2429 ($\pm$0.0515)	0.1527	13.5113 ($\pm$0.0515)	0.0878
	CV Log-Normal	13.4221 ($\pm$0.0194)	0.0181	13.4983 ($\pm$0.0194)	0.0124



(a) $N \longmapsto |I_0 - \widehat{\widetilde{I}}_M^{\lambda,N}|$ (▼), $N \longmapsto |I_0 - \widehat{\widetilde{\widetilde{I}}}_M^{\lambda,N}|$ (●) and the Crude Monte Carlo estimator (■) with their associated confidence interval at 95%.



(b) $N \longmapsto MSE(\widehat{I}_M)$ (■), $N \longmapsto MSE(\widehat{\widetilde{I}}_M^{\lambda,N})$ (▼) and $N \longmapsto MSE(\widehat{\widetilde{\widetilde{I}}}_M^{\lambda,N})$ (●).

Figure: $n = 128$, $M = 1e4$.

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Thank you for your attention!